Turbulence

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Master 2 DET

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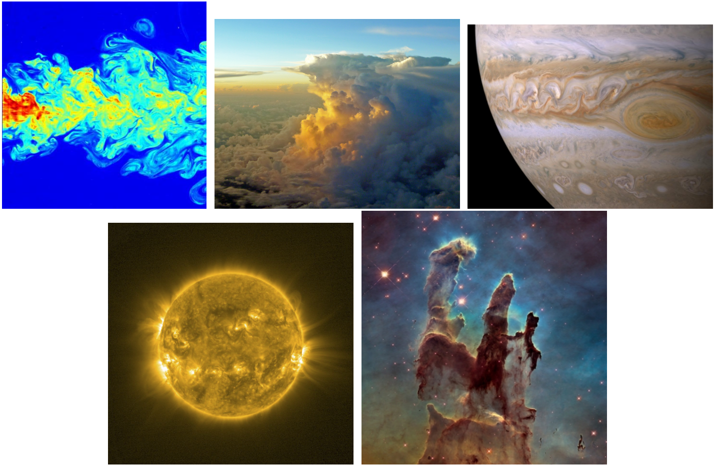
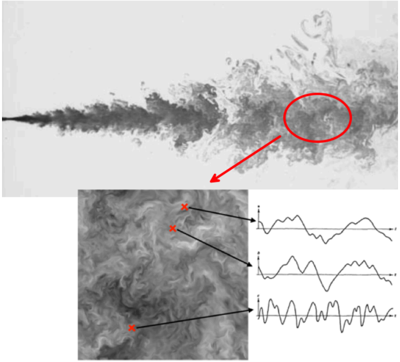
Outline

Introduction

Introduction

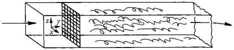


Turbulent fluctuations

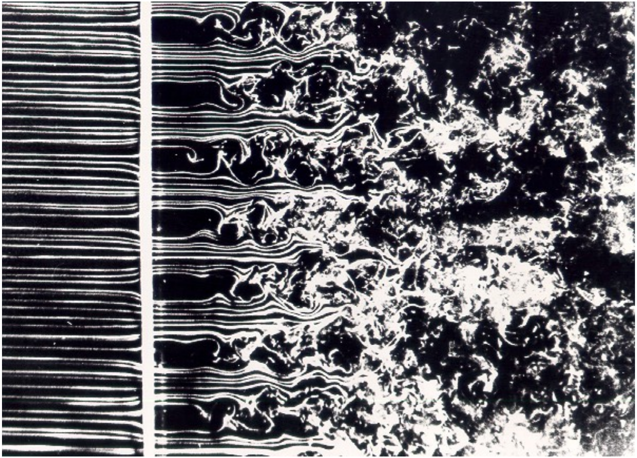


- Large range of scales
- Self organization
- Universality ?

Turbulent fluctuations



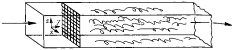
Generation of turbulence by a grid in a wind tunnel



Turbulent flow behind a grid, T. Corke & H. Nagib

Large range of scale in tubrbulence

Turbulent fluctuations



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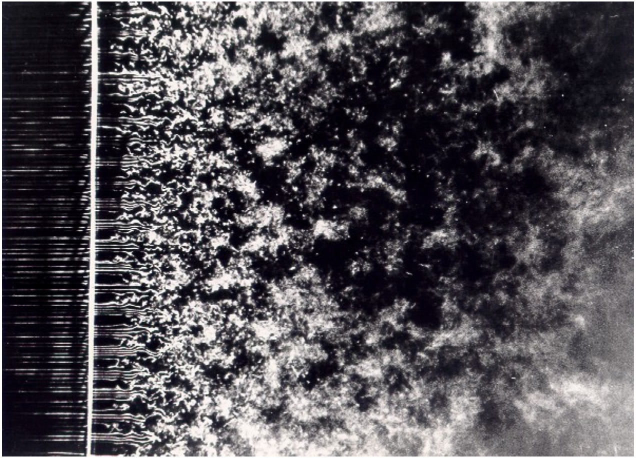

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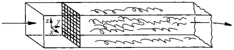

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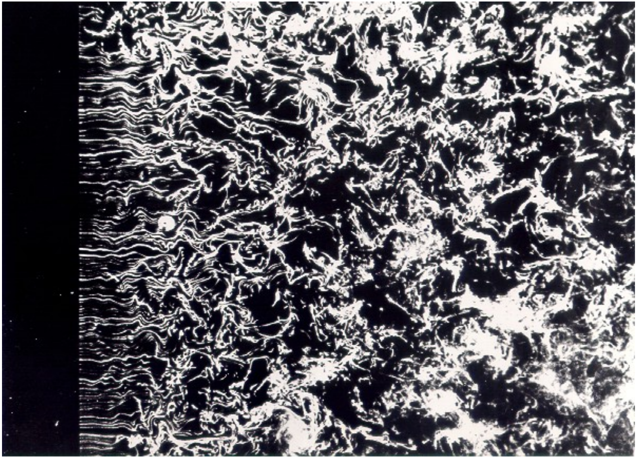

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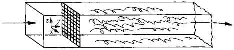

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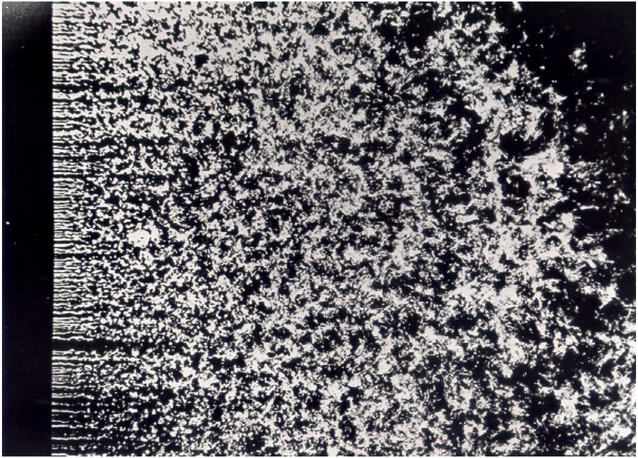
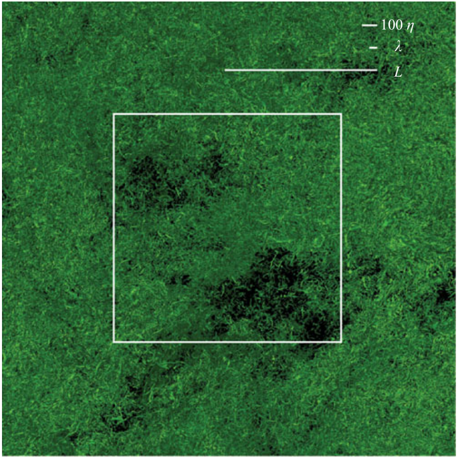


Turbulent flow behind a grid, T. Corke & H. Nagib

Large range of scale in tubrbulence

Turbulent fluctuations

Direct Numerical Simulation of a homogenous and isotropic turbulent flow



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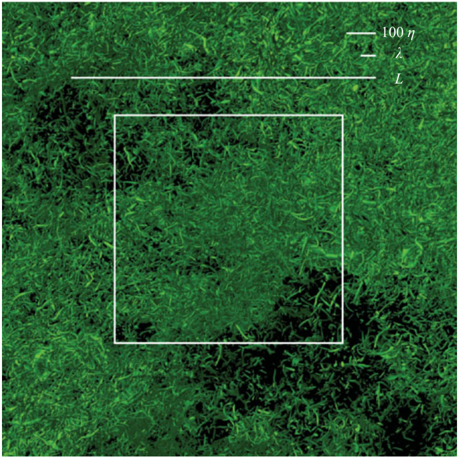

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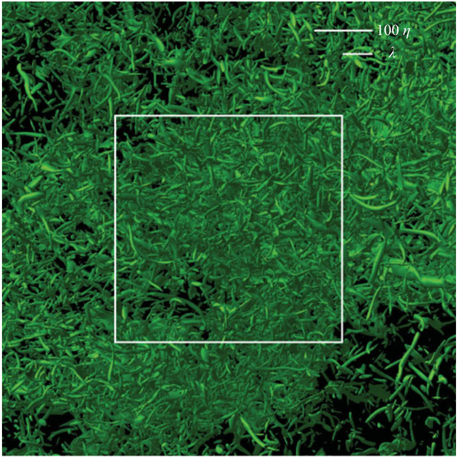

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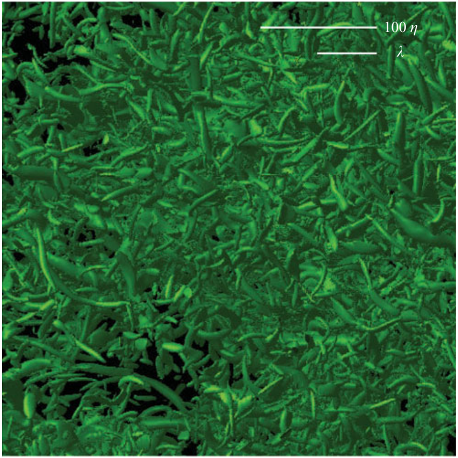

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Large range of scale in tubrbulence

Introduction

- Turbulent flows are everywhere
- inertia effect are dominating $\Rightarrow Re \gg 1$
- $\Rightarrow$ Non-linear $\Rightarrow$ unpredictable $\Rightarrow$ random
- Large range of scale in tubrbulence (cascade)
- Dissipative phenomena (viscosity are important even $Re \gg 1$)

Open problem

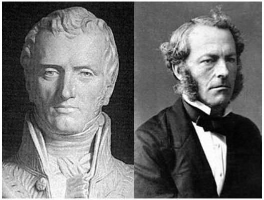
- Open problem of physic / Classical Mechanic
- No general turbulence theory
- issues in physics, engineering and applied mathematics

Contexte

Aeronautics, ground transportation, energy production, process engineering, geophysics, astrophysics . . .

Navier-Stokes equations

Very low Mach number (Ma) flows can described by the incompressible Naviers-Stokes equations



Navier(1785-1836): Introduce the viscosity in the Euler equation in 1823

Stokes (1819-1903): Modern form of the equation of momentum conservation in 1845

$$\underbrace{\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}.\nabla \boldsymbol{u}}_{\text{acceleration}} = \underbrace{-\frac{1}{\rho}\nabla P}_{\text{pressure forces}} + \underbrace{\nu \nabla^2 \boldsymbol{u}}_{\text{viscous forces}} + \boldsymbol{f}$$

$$\nabla.\mathbf{u} = 0$$

The existence of solutions to these equations has not been demonstrated. But approximate solutions can be obtained by **numerical simulations**

Navier-Stokes equations

The pressure field is determined by the incompressibility condition:
Take the divergence of the Navier-Stokes equation:

$$\frac{\partial}{\partial x_i}\left\{\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu\frac{\partial^2 u_i}{\partial x_j \partial x_j}\right\} \quad \text{with } \rho=\text{cst and } \frac{\partial u_i}{\partial x_i}=0$$

One obtains a Poisson equation for the pressure:

$$\nabla^2 p=-\rho\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}$$

Which has the following formal solution (if $p=\text{cst}$ as $r\to\infty$):

$$p(\mathbf{x})=\frac{\rho}{4\pi}\int_{\mathcal{V}}\frac{\partial u_i}{\partial y_j}\frac{\partial u_j}{\partial y_i}(\mathbf{y})\frac{d^3\mathbf{y}}{r} \quad \text{with } r=x-y$$

all points y of the flow domain contribute to the pressure at a given point x:
Non-local relation between **pressure** and **velocity**

⇒ ∇P Non-local and non-linear

⇒ Naviers-Stokes equations are an intégro-différential system.

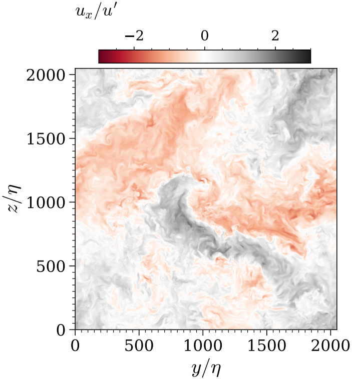
Reynolds Number

The Reynolds number characterizes the importance of inertial effects compared to viscous effect:

$$Re = \frac{O(\boldsymbol{u}.\nabla \boldsymbol{u})}{O(\nu \Delta \boldsymbol{u})} = \frac{U' L}{\nu}$$

⚠This lecture: fully develop incompressible turbulence: $Ma \ll 1$ et $Re \gg 1$

At large Reynolds number, turbulent flows present large fluctuations and display complex structures



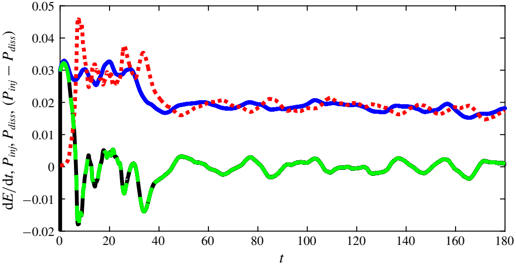
Order of magnitude of the fluctuations?

Turbulent Cascade

Energy, Injection, dissipation

Kinetic energy budget:
$E = \frac{1}{2}\int \rho \boldsymbol{u}^2 d^3x$:

$$\frac{dE}{dt} = P_{\text{inj}} - P_{\text{diss}}$$



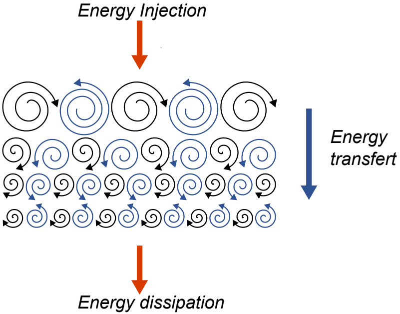
DNS, Dubrulle JFM 2019

- P_{inj} = Work of a volume force
Energy is injected at scale H
- $P_{\text{diss}} = \int d^3x\rho \underbrace{\frac{\nu}{2}\sum_{i,j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2}_{\varepsilon}$

⚠Dissipation is not taking place at the scale of injection

Kolmogorov theory

Turbulent Cascade



- Energy injection at $\ell = H$
- **Energy cascade**: Energy transfer towards smaller and smaller scales
- Energy dissipation at $\ell = \eta$

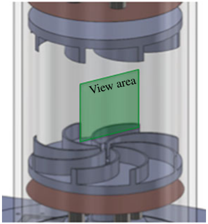
⚠Dissipation does not depend on viscosity (confirmed experimentally!)

It is call; sometimes: **anomalous dissipation** *remember* $\varepsilon = \frac{\nu}{2}\sum_{i,j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2$

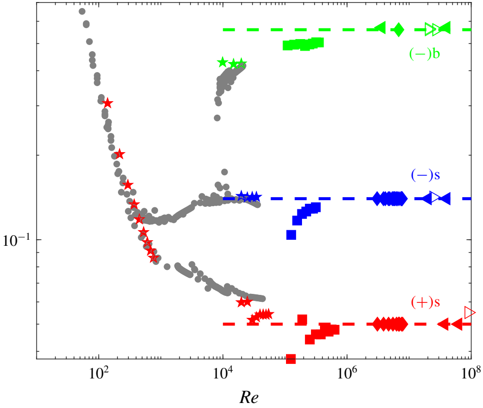
When viscosity is modified, the depth of the cascade is altered → small scales adapt to dissipate the energy injected on large scales

Turbulent Cascade

Experimental test of the Anomalous dissipation (0th Law of turbulence)



Normalized dissipation $\langle \varepsilon \rangle / L^2 \Omega^3$ vs Re



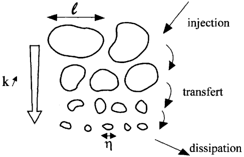
Dubrulle JFM 2019, Saint-Michel 2013

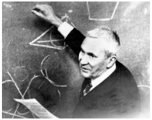
Average power (energy injection or energy dissipation) is independent of the viscosity

Turbulent Cascade

Kolmogorov laws

In 1941 Kolmogorov proposed an estimate of the intensity of motion at a given size ℓ and for the dissipative scale η.





Andreï Kolmogorov

(1903-1987)

- The intensity of motion at ℓ scale must be defined as a variation in velocity over distance ℓ. (No effect of global advection velocity on flow evolution, Galilean invariance).

- We consider scales much smaller than H ⇒ **quasi-homogeneity**: the statistical properties of the velocity field are invariant by translation.

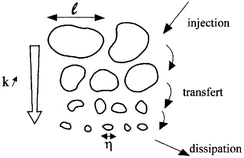
- E_{cin} et P_{diss} are extensive quantites → better to consider the energy density and dissipation density (i.e per unit of mass)

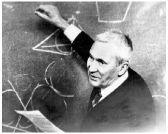
$$\langle \epsilon \rangle = \frac{\nu}{2} \langle \sum_{i,j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 \rangle \qquad [\varepsilon] = L^2 T^{-3} = W/kg$$

- For stationary conditions: $\langle \epsilon \rangle$ = injection rate = transfer rate = dissipation rate

Turbulent cascade

Kolmogorov law





Andreï Kolmogorov

(1903-1987)

- The kinetic energy of the eddies of size ℓ, δu_ℓ^2, is transferred to smaller and smaller structures.
- The kinetic energy transfer rate $\varepsilon(\ell)$ along the cascade is constant.

$$\varepsilon_\ell = \frac{\delta u_\ell^2}{\tau_\ell} = \frac{\delta u_\ell^3}{\ell} = \text{cst} = \langle\varepsilon\rangle \qquad \forall \ell$$

The characteristic velocity only depends on $\langle\varepsilon\rangle$ and ℓ

$$\delta u_\ell = (\langle\varepsilon\rangle\ell)^{1/3}$$

▸ No characteristic scale (power law) → self similarity of the flow
With these hypothesis, the flow depends on H only through $\langle\varepsilon\rangle$
→ Looking at a given scale, one cannot distinguish two turbulent flows with the same $\langle\varepsilon\rangle$ but different H.

Scale invariance

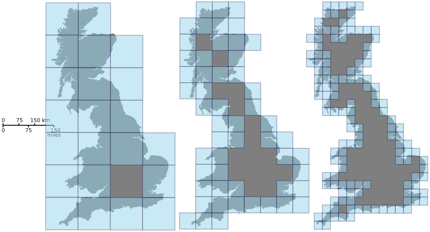
Scale invariance = system unchanged (i.e. invariant) when considering it at different scales

Scale-invariant object $f(x)$ must satisfy the condition:

$$f(\lambda x) = \lambda^{\alpha} f(x)$$

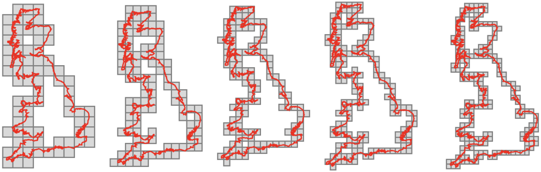
$f(x)$ is an homogenous function like $f(x) = c\ x^{\alpha}$

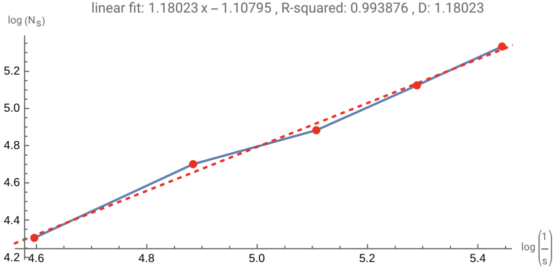
Scale-invariant object have no characteristic scale.
E.g. "fractal dimension"



⚠Physically their is always a cut-off a very large/small scales

Scale invariance

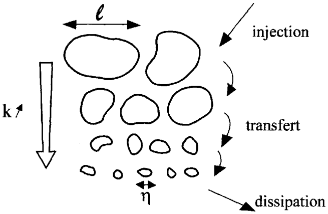




`https://demonstrations.wolfram.com/BoxCountingTheDimensionOfCoastlines/`

Turbulent cascade

Kolmogorov law



What is the size of the smallest structures of a turbulent flow?

At sufficiently small scales, the viscosity smooth the velocity difference, and the energy transfer towards smaller scales stops.

At the smallest scales, inertia and viscous effects are in balance:

$$Re_\eta = \frac{u_\eta \eta}{\nu} = 1$$

⇒ one obtains the Kolmogorov dissipative scales:

$$\eta = \langle \varepsilon \rangle^{-1/4} \nu^{3/4} \qquad \tau_\eta = \langle \varepsilon \rangle^{-1/2} \nu^{1/2}$$

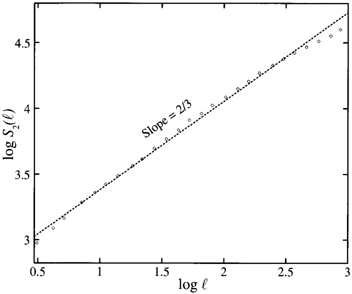
The Reynolds number ($Re_H = \frac{UH}{\nu}$) is a measure of the scale separation:

$$\frac{H}{\eta} \sim Re_H^{3/4} \qquad \frac{\tau_H}{\tau_\eta} \sim Re_H^{1/2}$$

The Kolmogorov theory

Confirmed experimentally !

$$\langle \delta_r u^2 \rangle \sim (\langle \varepsilon \rangle r)^{2/3} \quad \text{for } \eta \ll r \ll L$$



Gange & Hopfinger

Example: turbulent Dispersion

Richardson's problem

Consider the evolution of the distance between 2 particles in a turbulent flow with an initial separation of ℓ_0

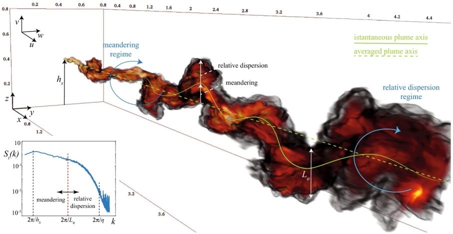
With the Kolmogorov relation

$$\frac{d\ell}{dt} = u(\ell) = (\langle \varepsilon \rangle \ell)^{1/3}$$

After integration:

$$\ell^{2/3} - \ell_0^{2/3} \sim \varepsilon^{1/3} t$$

Then we estimate that the diameter of a cloud of pollutant grows as $\ell(t) \sim \varepsilon^{1/2} t^{3/2}$

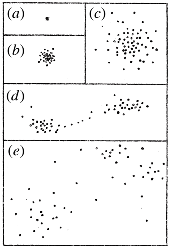


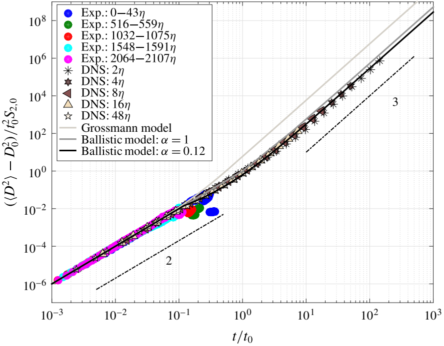
Cassiani et al 2020

Example: turbulent Dispersion

Richardson's problem

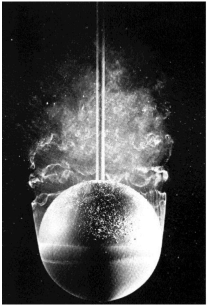
test of the $\ell(t) \sim \varepsilon^{1/2} t^{3/2}$ relation





Bourgoin JFM 2016

Example: Limit velocity of a fall



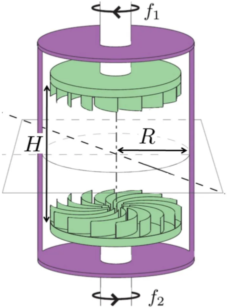
An objet of size ℓ falls and reach a stationary velocity u when the drag force balance the gravitational force.

- In the laminar case the drag force is proportional to the viscosity: $F \sim \nu\rho\ell u$
We get: $u \sim mg/\nu\rho\ell$

- In the turbulent case, the Kolmogorov law ($\varepsilon \sim u^3/\ell$) indicates that the power is independent of the viscosity and varies as the cube of the velocity. As the power is the product of the force with the velocity, the force must proportional to the square of the velocity $F \sim \rho\ell^2 u^2$ and so $u \sim \sqrt{mg/\rho\ell^2}$

- Give an estimate of the velocity of a parachutist (with open/closed parachute) and compare with a laminar estimation

von Karman flow



- Flow driven by two counter-rotating disc at 20Hz
- Diameter: 10 cm
- 2kg of water
- Estimate the power and the Reynolds number
- To increase the Reynolds number is it better to multiply the rotation speed by 10 or use mercury? $(\nu_{Hg} = 0.1\nu_{H_20}\ ;\ \rho_{Hg} = 13.6\rho_{H_20})$

Refreshing on statistics

Refreshing on statistics

- Average
- Moments variance, skewness, flatness
- central limit theorem
- Probability density function

Refreshing on statistics

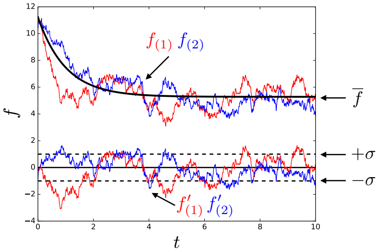
random processes

The ensemble average of quantity f is the average over an ensemble of realizations of $f_{(i)}$:

$$\langle f\rangle(\mathbf{x},t) = \lim_{N\to\infty}\frac{1}{N}\sum_{i}^{N} f_{(i)}(\mathbf{x},t)$$

The fluctuation around the mean value is:

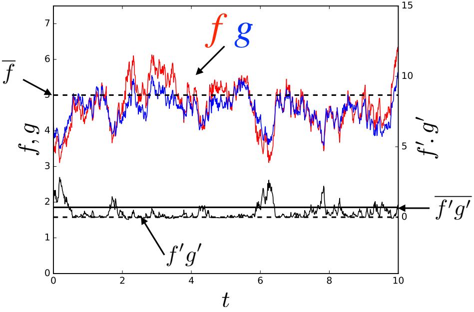
$$f'_{(i)} = f_{(i)} - \langle f\rangle$$



Refreshing on statistics

Some related definitions:

- Variance: $\sigma^2 = \langle f'^2 \rangle$
- Standard deviation: $\sigma = \sqrt{\langle f'^2 \rangle}$
- Correlation (or covariance) : $R = \langle f' g' \rangle$



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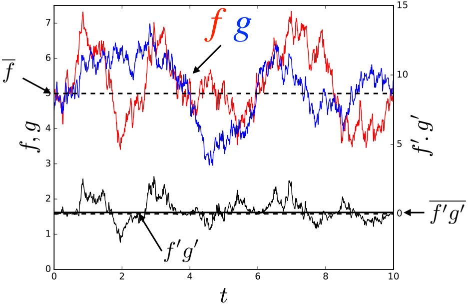

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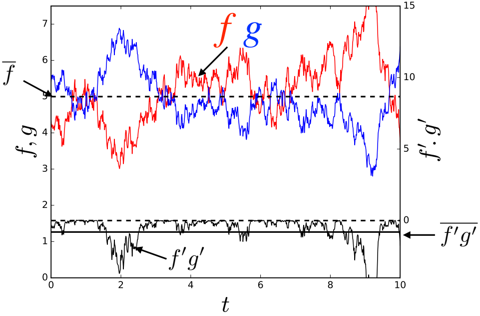

Refreshing on statistics

Some properties:

- The average is a linear operator:

$$\langle\langle f\rangle\rangle = \langle f\rangle \ \Rightarrow \langle f'\rangle = 0$$

$$\langle \alpha f\rangle = \alpha \ \langle f\rangle$$

$$\langle f+g\rangle = \langle f\rangle + \langle g\rangle$$

- the averaging commutes with spatial and temporal derivatives:

$$\frac{\partial\langle f\rangle}{\partial x} = \langle\frac{\partial f}{\partial x}\rangle \quad ; \quad \frac{\partial\langle f\rangle}{\partial t} = \langle\frac{\partial f}{\partial t}\rangle$$

- average of a product :

$$\langle f g\rangle = \langle f\rangle\langle g\rangle + \langle f' g'\rangle$$

Refreshing on statistics

A random variable, U, is characterized by its probability density function

- A random field, such as velocity, is far more complex than a random variable.
  - Random variable U_1, U_2, U_3
  - Random function of time $U(t)$
  - Random field of space and time $\mathbf{U}(\mathbf{x}, t)$

Refreshing on statistics

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event

For a random variable, we define an event, e.g.:

- $A = \{U < 1m/s\}$
- $B = \{U < V_b\}$
- $C = \{V_a \leq U < V_b\}$

Probability

the probability of the event:

$$p = P(B) = P\{U < V_b\}$$

Here p describes the likelihood of the event: $0 \leq p \leq 1$

Statistic refreshing

- Composition of probability: **AND**

*If two events are **independent**, the probability are mulipl ied*

Independent: The results obtained for one event as no influence on the other.

Generally, for two **independent** events E_1 and E_2 combined by a logical **"AND"** and describe by probability laws P_1 and P_2, we have

$$P(E_1 \text{ AND } E_2) = P_1(E_1)P_2(E_2)$$

- Composition of probability: **OR**

*If two events are **mutually exclusive**, the probability to obtain one event or the other is the sum of the individual probabilities*:

$$P(E_1 \text{ OR } E_2) = P_1(E_1) + P_2(E_2)$$

Statistic refreshing

Probability Density Function **PDF**

$P(x < X < x+h)$ is the probability to obtain a result X between x and $x+h$.

Splitting the interval $[x, x+h]$ in two, we can naturally write (with the "OR" composition law):

$$P(x < X < x+h) = P(x < X < x+h/2) + P(x+h/2 < X < x+h)$$

Dividing the interval in N with $N \rightarrow \infty$ we get:

$$P(x < X < x+h) = \int_x^{x+h} p_X(x) dx$$

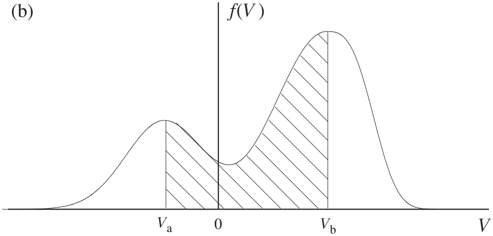
taking an infinitly small intervall $h = dx$, we write:

$$P(x < X < x+dx) = p_X(x) dx$$

The function $p_X(x)$ is the probability density of X.
(It is the probability per "unit of X")

Statistic refreshing

Probability Density Function **PDF**



Properties

- $p_X(x) \geq 0$;
- $p_X(-\infty) = p_X(\infty) = 0$;
- $\int_{-\infty}^{\infty} p_X(x)dx = 1$ the PDF is normalized

Question:?

- What is the unit of p_X ?

Statistic refreshing
PDF and change of variable

Consider the change of variable $Y = f(X)$
What is the PDF of Y (assuming the pdf of X is known) ?

$$P(x < X < x + dx) = P(f(x) < f(X) < f(x + dx)) \quad (1)$$

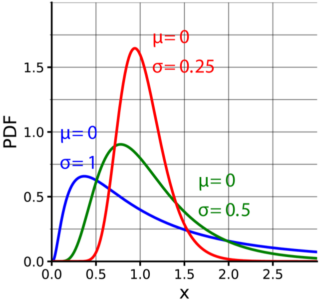
$$P_X(x)dx = P_Y(y)dy \quad (2)$$

$$P_Y(y) = P_X(x(y))(dy/dx)^{-1} \quad (3)$$

- Example: **"Log-normal" variable**

X is a Gaussian variable:
$P_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$
Get the PDF of a "Log-normal" variable
$Y = \exp X$

$$P_Y(y) = \frac{1}{y\sqrt{2\pi\sigma^2}} \exp(-\frac{(\ln(y)-\mu)^2}{2\sigma^2})$$



Refreshing on statistics
Distribution density function

Average and moments

- The average:
  $\langle X\rangle = \int_{-\infty}^{\infty} x p_X(x) dx$
  (sum weighted by probability density)

Refreshing on statistics
Distribution density function

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Refreshing on statistics
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Refreshing on statistics
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Refreshing on statistics
Distribution density function

Average and moments

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- Standard deviation: $\sigma_X = \sqrt{\sigma_X^2} = \sqrt{\langle X'^2 \rangle}$

Refreshing on statistics
Distribution density function

Average and moments

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- Standard deviation: $\sigma_X = \sqrt{\sigma_X^2} = \sqrt{\langle X'^2\rangle}$
- nth-order centered moment: $\langle X'^n\rangle = \int_{-\infty}^{\infty} (x - \langle X\rangle)^n p_X(x) dx$

Refreshing on statistics
Distribution density function

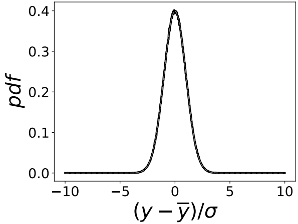
Central limit theorem

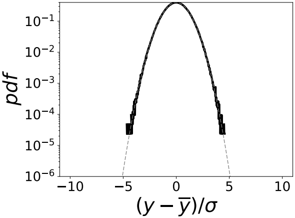
The sum of a large number of independent random variables presents a Gaussian distribution function

$$y = \sum_{i=1}^{N} x_i$$

$\langle y \rangle = N \langle x_i \rangle$; $\sigma_y^2 = \langle y'^2 \rangle = N \langle x_i'^2 \rangle$

And for N large : $f(y) = \dfrac{1}{\sqrt{2\pi\sigma^2}} \exp\left(- \dfrac{\left(y - \langle y \rangle\right)^2}{2\sigma^2} \right)$

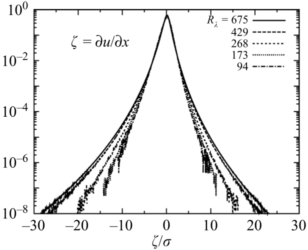




PDF of velocity fluctuations

In turbulent flow, if the volume for calculating statistics is **important**, $\ell \gg L_{int}$ (or the period is large), we can consider a juxtaposition of independent events.
⇒ velocity has a **gaussian** distribution

But the distribution of velocity increments $\delta u_{\parallel}(\vec{r})$ for distances $r < L_{int}$ is not Gaussian: correlation in the velocity field
⇒ the velocity field is not **gaussian** ⇒ **Intermittency**



Ishihara, 2007

Statistical description

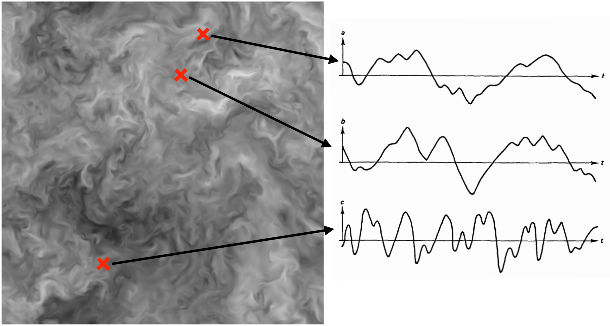
Statistical description of the turbulence

Correlation functions

Turbulence can be statistically describe with the correlation functions:

$$\mathscr{R}_{i\,j\ldots}(\boldsymbol{x_1}, t_1;\ \boldsymbol{x_2}, t_2;\ \ldots) = \langle u_i(\boldsymbol{x_1}, t_1)\ u_j(\boldsymbol{x_2}, t_2)\ldots\rangle$$

The complete description of the turbulence requires the knowledge of all the correlation functions



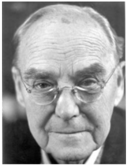
We also consider the structure function $\langle \delta_r u^n \rangle = \langle (u(x+r) - u(x))^n \rangle$ (another kind of correlation function)

Statistical description of the turbulence

Two-points Correlation

The simplest and most important is the **two-points correlation** function
(Introduced by G. I. Taylor in 1935: beginning of the modern statistical approach):

$$\mathscr{R}_{ij}(\boldsymbol{x_1}, \boldsymbol{r}) = \langle u'_i(\boldsymbol{x_1}, t)\, u'_j(\underbrace{\boldsymbol{x_1} + \boldsymbol{r}}_{\boldsymbol{x_2}}, t)\rangle$$



Sir Geoffrey Ingram Taylor

(1886-1975)

⚠It is related to the energy of flow carried by the "structures" of size r

Two-points Correlation
Homogeneity

For statistically homogenous flows, the two-points correlation tensor only depends on the $\boldsymbol{r}$ vector separating the two points:

$$\mathscr{R}_{ij}(\boldsymbol{r}) = \langle u_i(\boldsymbol{x},t)\, u_j(\boldsymbol{x}+\boldsymbol{r},t)\rangle$$

- ▸ Homogenous means invariance by translation in space

$$\begin{aligned}\mathscr{R}_{ij}(\boldsymbol{x_1},\boldsymbol{x_2}) &= \mathscr{R}_{ij}(\boldsymbol{x_1}-\boldsymbol{x_1},\boldsymbol{x_2}-\boldsymbol{x_1}) = \mathscr{R}_{ij}(0,\boldsymbol{r}) = \mathscr{R}_{ij}(\boldsymbol{r})\\ &= \mathscr{R}_{ij}(\boldsymbol{x_1}-\boldsymbol{x_2},\boldsymbol{x_2}-\boldsymbol{x_2}) = \mathscr{R}_{ij}(-\boldsymbol{r},0) = \mathscr{R}_{ij}(-\boldsymbol{r})\end{aligned}$$

$\mathscr{R}_{ij}(\boldsymbol{r}) = \mathscr{R}_{ij}(-\boldsymbol{r})$ is a pair function (and it is maximum at $\boldsymbol{r}=0$)

- ▸ ⚠With homogeneity and Galilean invariance, we can consider $u_i(\boldsymbol{x},t) = u'_i(\boldsymbol{x},t)$ in the definition of $\mathscr{R}_{ij}$.

Two-points Correlation
Incompressibility

The incompressibility condition of the flow $(\nabla \cdot \boldsymbol{u} = 0)$ imposes that

$$\begin{aligned}
\frac{\partial \mathscr{R}_{ij}(\boldsymbol{r})}{\partial r_j} &= \frac{\partial \langle u_i(\boldsymbol{x_1}, t)\, u_j(\boldsymbol{x_2}, t)\rangle}{\partial r_j} \\
&= \frac{\partial \langle u_i(\boldsymbol{x_1}, t)\, u_j(\boldsymbol{x_2}, t)\rangle}{\partial x_{2,j}} \\
&= \langle \frac{\partial u_i(\boldsymbol{x_1}, t)\, u_j(\boldsymbol{x_2}, t)}{\partial x_{2,j}} \rangle \\
&= \langle u_i(\boldsymbol{x_1}, t)\, \underbrace{\frac{\partial u_j(\boldsymbol{x_2}, t)}{\partial x_{2,j}}}_{=0} \rangle \\
&= 0
\end{aligned}$$

and likewise $\dfrac{\partial \mathscr{R}_{ij}(\boldsymbol{r})}{\partial r_i} = 0$

- Show it also using definition of derivative
- Physical interpretation ?

Two-points Correlation

Isotropy

If, in addition, we assume **statistical isotropy** of the flow (i.e. any statistical quantity looks the same if we change the orientation of the frame), the two-point correlation tensor can be express in the following way:

$$\mathscr{R}_{ij}(\boldsymbol{r}) = A(r)\delta_{ij} + B(r) r_i r_j$$

- Because of isotropy the only vector that can appear is $\boldsymbol{r}$.

- The two functions A and B only depends on the distance $r = |\boldsymbol{r}|$ between the two points.

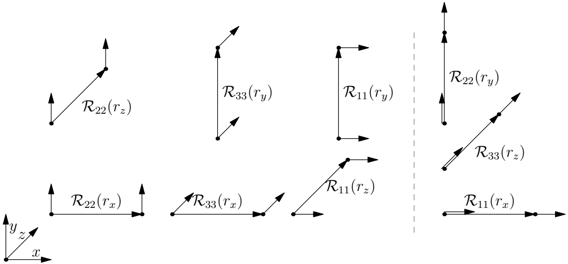
- ⚠With isotropy we have : $\langle u_x^2 \rangle = \langle u_y^2 \rangle = \langle u_z^2 \rangle = \frac{2}{3} K$

We introduce the **longitudinal and transversal correlation functions** to express A and B

Two-points Correlation

Longitudinal and transversal correlation functions

- The velocity vectors are parallel or perpendicular to the separation vector



Longitudinal correlation function

$$\mathscr{R}_{xx}(r\boldsymbol{e_x}) = \langle u_x^2 \rangle \; f(r)$$

Transversal correlation function

$$\mathscr{R}_{yy}(r\boldsymbol{e_x}) = \langle u_x^2 \rangle \; g(r)$$

Show that $B(r) = \langle u_x^2 \rangle \; (f(r) - g(r))/r^2$ and $A(r) = \langle u_x^2 \rangle \; g(r)$

Two-points Correlation
Isotropy

- With isotropy we show that $A(r) = \langle u_x^2 \rangle \ (f(r) - g(r))/r^2$ and $B(r) = \langle u_x^2 \rangle \ g(r)$ giving:

$$\mathscr{R}_{ij}(\boldsymbol{r}) = \langle u_x^2 \rangle \left((f(r) - g(r)) \frac{r_i r_j}{r^2} + g(r)\delta_{ij} \right)$$

- And using the incompressibility condition we have (noting $f' = df/dr$): $g = f + \frac{r}{2} f'$

$\Rightarrow$ When the flow is isotropic and incompressible the **longitudinal correlation** function alone fully determine all the two-point correlation:

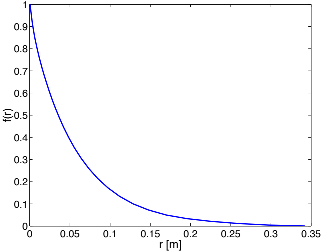
$$\mathscr{R}_{ij}(\boldsymbol{r}) = \langle u_x^2 \ \rangle \left(f(r)\delta_{ij} + \frac{r}{2} f' \left(\delta_{ij} - \frac{r_i r_j}{r^2} \right) \right)$$

- Note also that $\mathscr{R}_{ii}(\boldsymbol{r}) = \mathscr{R}_{ij}(\boldsymbol{r})\delta_{ij} = \langle u_x^2 \rangle \left(3f + rf'\right)$

$\Rightarrow$ The **longitudinal correlation** function plays a key role in describing the structure of turbulent flows

The characteristic scales of turbulence
Integral scale

- At $r = 0$, the longitudinal correlation function: $f(0) = 1$
- and forlarge enough r , $u_i(\boldsymbol{x})$ and $u_i(\boldsymbol{x}+\boldsymbol{r})$ becomes decorrelated: $f(r \to \infty) = 0$



→ The progressive decorrelation of velocity results in a correlation scale.
Macroscale or longitudinal **integral scale** of the flow:

$$L_{int} = \int_0^{\infty} f(r)dr$$

- Characteristic scale of the size of the largest turbulent structures in the flow.
- Also scale of mean velocity field inhomogeneity.
- What about the transversal integral scale ?

The characteristic scales of turbulence

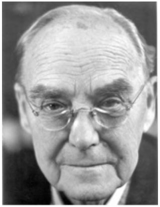
Taylor micro scale

We defined a micro-scale from the development of f around $r = 0$:

$$f(r) = \underbrace{f(0)}_{=1} + \underbrace{r \frac{df}{dr}\Big|_{r=0}}_{=0} + \frac{r^2}{2} \frac{d^2 f}{dr^2}\Big|_{r=0} + \dots$$

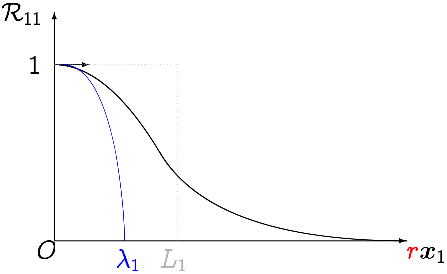
homogéniété

f est pair



Sir Geoffrey Ingram Taylor

(1886-1975)



- $f(r) \approx 1 - \frac{1}{2}\left(\frac{r}{\lambda}\right)^2 + O(r^4)$ with $\frac{1}{\lambda^2} = -f''$
- The micro-scale λ is usefull to characterize the intensity of velocity gradient
- what about the transversal micro scale ?

The characteristic scales of turbulence

Taylor micro scale

The micro-scale λ is usefull to characterize the intensity of velocity gradient:

$$\langle\left(\frac{\partial u_x}{\partial x}\right)^2\rangle = \langle\lim_{h\to 0}\frac{(u_x(\boldsymbol{x}+h\boldsymbol{e_x}) - u_x(\boldsymbol{x}))^2}{h^2}\rangle = \lim_{r\to 0}\frac{2\langle u_x^2\rangle\left(1-f(r)\right)}{r^2} = \frac{\langle u_x^2\rangle}{\lambda^2}$$

And so one can estimate the (average) dissipation rate $\varepsilon = 2\nu S'_{ij}S'_{ij}$.

For a **isotropic** flow, we have: $\langle\varepsilon\rangle = 15\nu\langle\left(\frac{\partial u_x}{\partial x}\right)^2\rangle$ and $\frac{3}{2}\langle u_x^2\rangle = \langle\mathscr{K}\rangle$, thus:

$$\langle\varepsilon\rangle = 10\nu\frac{\langle\mathscr{K}\rangle}{\lambda^2}$$

⚠Although λ characterize the dissipation in the flow, it does not represent a group of turbulent structures for which dissipation is significant

⚠The smallest scale of turbulence is the dissipative Kolmogorov scale $\eta = (\nu^3/\varepsilon)^{1/4}$

The characteristic scales of turbulence

Taylor micro scale

Details of the calculation:

Assuming isotropy $\langle \left(\frac{\partial u_i}{\partial x_j}\right)^2 \rangle = 3\langle \left(\frac{\partial u_x}{\partial x}\right)^2 \rangle + 6\langle \left(\frac{\partial u_x}{\partial y}\right)^2 \rangle$

$$\langle \left(\frac{\partial u_x}{\partial x}\right)^2 \rangle = \lim_{h\to 0} \frac{2\left(\langle u_x^2\rangle - R_{xx}(h\boldsymbol{e_x})\right)}{h^2} = \lim_{h\to 0} \frac{2\langle u_x^2\rangle\left(1-f(h)\right)}{h^2}$$

and similarly $\langle \left(\frac{\partial u_x}{\partial y}\right)^2 \rangle = \lim_{h\to 0} \frac{2\langle u_x^2\rangle\left(1-g(h)\right)}{h^2}$

We have $g(r) = f + \frac{r}{2}f'$ and with the Taylor expansion $f'(0) = -\frac{r}{\lambda^2}$

combining all these gives $\langle \varepsilon \rangle = 15\nu \langle \left(\frac{\partial u_x}{\partial x}\right)^2 \rangle = 15\nu \frac{\langle u_x^2\rangle}{\lambda^2}$

The characteristic scales of turbulence

The taylor scale Reynolds number

We obtain a relationship between λ and L_{int} from the simplified energy balance ($P_{\text{inj}} = P_{\text{diss}}$):

$$\frac{u'^3}{L_{int}} \sim \nu \left(\frac{u'}{\lambda} \right)^2$$

$$\begin{aligned} \frac{L_{int}}{\lambda} &= \frac{u'\lambda}{\nu} = Re_{\lambda} \\ &= \left(\frac{u' L_{int}}{\nu} \right)^{1/2} = Re_{local}^{1/2} \end{aligned}$$

Re_{λ} is a measure of the **separation** of scales

Characteristic scales of turbulence
Velocity increments - Scale separation

Let's introduce the velocity increments:

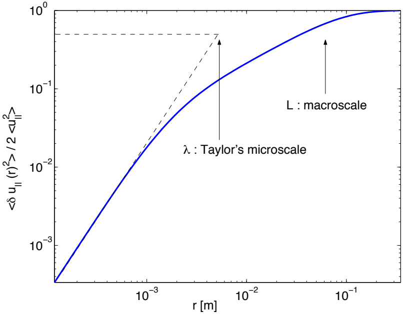
$$\delta u_x(r) = \delta_r u = u_x(\boldsymbol{x}, t) - u_x(\boldsymbol{x} + r\boldsymbol{e}_x, t)$$

- It describe the distribution of energy across scales.
- Homogeneity: the averaged velocity increment is zero
- Variance of the velocity increments is: $\langle(\delta_r u)^2\rangle = 2\langle u_x^2\rangle\left(1 - f\right)$

- $\dfrac{\langle(\delta_r u)^2\rangle}{2\langle u_x^2\rangle} \approx \dfrac{1}{2}\left(\dfrac{r}{\lambda}\right)^2$ pour $r \to 0$
- $\dfrac{\langle(\delta_r u)^2\rangle}{2\langle u_x^2\rangle} \approx 1$ pour $r \geq L$

Three ranges:

- Energy scales
- Inertial scales
- Dissipative scales



The Kolmogorov theory[1]

$$\begin{aligned}\langle \delta_r u^2 \rangle &= fct(r, \nu, \langle \varepsilon \rangle, L, \dots) \\ &= (\langle \varepsilon \rangle r)^{2/3} \Phi(r/L, r/\eta)\end{aligned}$$

1^{st} hypothesis = Local isotropy

For high enough Reynolds number, turbulence is **universal**

For $r \ll L$: velocity increment statistics are independent of the specific forcing mechanism $\left(\lim_{r/L \to 0} \Phi(r/L, r/\eta) = \phi(r/\eta)\right)$

$$\langle \delta_r u^2 \rangle = (\langle \varepsilon \rangle r)^{2/3} \; \Phi(r/\eta)$$

2^{nd} hypothesis = Inertial range

For $r \gg \eta$: velocity increment statistics become independent of the viscosity

$\lim_{r/\eta \to \infty} \phi(r/\eta) = cst \Rightarrow$ *complete similarity* (Barenblatt)

$$\langle \delta_r u^2 \rangle \sim (\langle \varepsilon \rangle r)^{2/3} \quad \eta \ll r \ll L$$

- the only parameter is the energy transfer rate → **Energy cascade**
- No characteristic scale (power law)
  → self similarity of the flow

BUT . . .

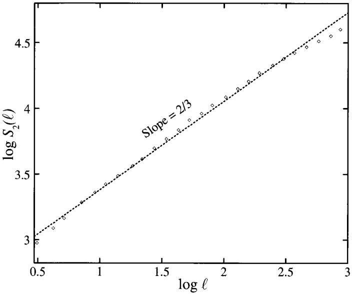
---

[1] *A. N. Kolmogorov. Dokl. Akad. Nauk SSSR, 434:9–13, 1941.*

The Kolmogorov theory

Confirmed experimentally !

$$\langle \delta_r u^2 \rangle \sim (\langle \varepsilon \rangle r)^{2/3} \quad \text{for } \eta \ll r \ll L$$



Gange & Hopfinger

The Kolmogorov theory
Velocity spectra

Confirmed experimentally !

Energy spectrum = energy density per wavenumber
$E(k) \sim \langle \delta_r u^2 \rangle / k$ with $k \sim 1/r$

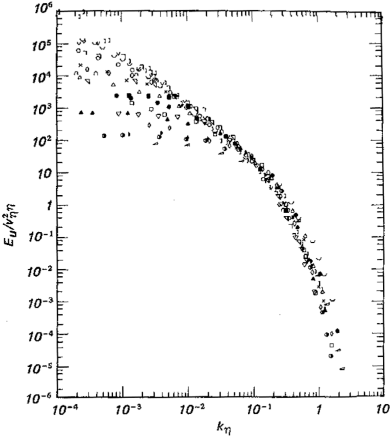
Universal energy spectrum $E(k)$ at high wavenumbers:

$$E(k) = \langle \varepsilon \rangle^{2/3} k^{-5/3} \phi(k\eta)$$

Kolmogorov spectra[2]: inertial range ($k\eta \gg 1$)

$$E(k) = C_K \, \varepsilon^{2/3} \, k^{-5/3}$$

and $C_K \approx 1.5$



from Monin & Yaglom

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[2] Actually proposed by Oboukov (1941) and independently by several other authors: Onsager (1945), Heisenberg (1948), Orszag . . .)

The Karman-Howarth equation and the 4/5 law

The Karman-Howarth equation

To be completed . . .

The Kolmogorov theory

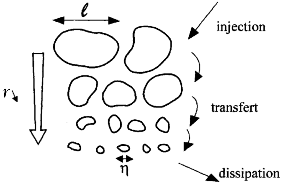
the "4/5 law": Skewness of the velocity increments

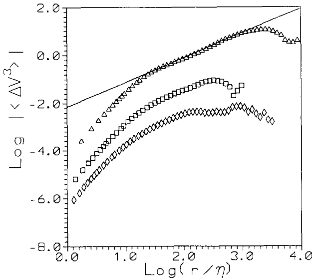
In addition to the previous relations, Kolmogorov established[3] an **exact relation** (Navier-Stokes equation + stat. homogeneity).

$$\langle \delta_r u^3 \rangle = -\frac{4}{5} \langle \varepsilon \rangle r + 5\nu \frac{d}{dr} \langle \delta_r u^2 \rangle$$

(for details see Karman-Howard-Monin equation or Lin equation)

Energy transfer in the cascade
⇒ Skewness of the velocity increments





Benzi et al, Physica D, 1995

Physically: transfer from the large scales to the smaller ones by stretching of vortices

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[3] *A. N. Kolmogorov. Dokl. Akad. Nauk SSSR, 32:1, 1941.*

The Kolmogorov theory

higher moments ?

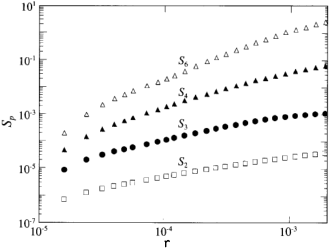
Generalization to higher moments

$$\langle|\delta_r u|^p\rangle \sim (\langle\varepsilon\rangle r)^{p/3} \quad \eta \ll r \ll L$$

- Empirical agreement for $p = 2$
- Exact for $p = 3$ (the Karman-Howarth eq.)
- But it **fails** for $p > 3$: $\langle|\delta_r u|^p\rangle$ behaves as power laws in the inertial range but with exponents the deviate from $p/3$

Structure function of order p vs r

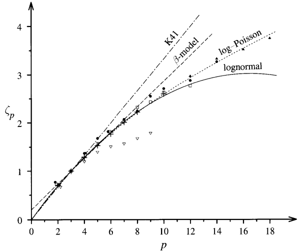
$$S_p = \langle|\delta_r u|^p\rangle$$



Mauer et al. 1994

Exponent of the structure function vs p

$$\langle|\delta_r u|^p\rangle = v_0^p\,(r/\ell_0)^{\zeta_p}$$



Frisch 1995

Intermittency

The Kolmogorov theory

higher moments ?

If the generalization works $\langle|\delta_r u|^p\rangle \sim (\langle\varepsilon\rangle r)^{p/3} \quad \eta \ll r \ll L$
it would imply gaussian distribution for $\delta_r u$ with variance: $\sigma^2 \sim (\langle\varepsilon\rangle r)^{2/3}$
(NB: for gaussian variables: $\langle x^p\rangle = \sigma^p (p-1)!!$)

But the velocity fields is **not gaussian**: **Skewness** + **Intermittency**

PDF of velocity increments

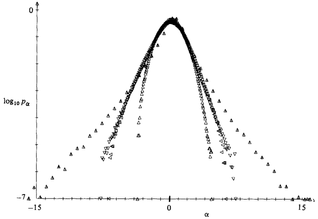
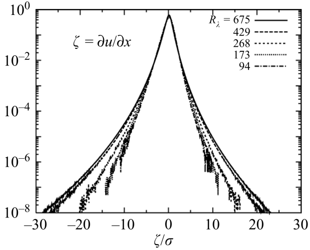


FIGURE 1. Probability density functions in the axisymmetric jet at $R_\lambda = 536$ of u and Δu normalized by their respective standard deviations, $\alpha = \Delta u/\langle\Delta u)^2\rangle^{\frac{1}{2}}$: $\triangle$, $r = 0.6$ mm $= 3.5\eta$; $\bigtriangledown$, 7.7 mm; $\triangleleft$, 17.2 mm. $\triangle$, $\alpha = u/\langle u^2\rangle^{\frac{1}{2}}$.

Anselmet et al., JFM 1984

PDF of velocity gradient

$$\partial_x u_x = \lim_{r\to 0} \delta_r u/r$$



Ishihara, JFM 2007

The Kolmogorov theory
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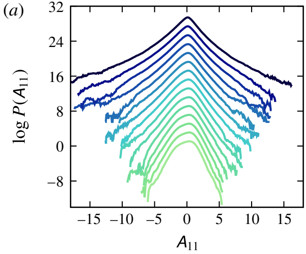
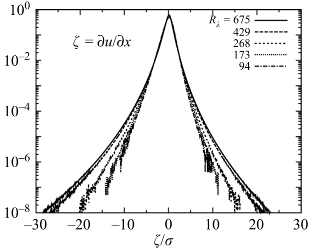


Pereira et al., 2018

PDF of velocity gradient

$$\partial_x u_x = \lim_{r\to 0} \delta_r u/r$$



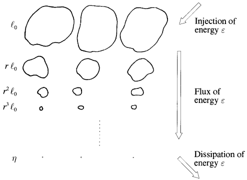
Ishihara, JFM 2007

Fractals and multifractals

The β model

We can explain the deviation from the dimensional analysis:

- Suppose the volume fraction ϕ_l of "active" eddies of size ℓ decreases as a power of ℓ:



$$\phi_l = \beta^n = \beta^{ln(\ell/\ell_0)/\ln\lambda} = \left(\frac{\ell}{\ell_0}\right)^{3-D}$$

with $3-D = \ln\beta/\ln\lambda$

- We keep the cascade picture (with $\tau_l = \ell/v_l$)

$$\varepsilon_\ell \sim \frac{E_L}{\tau_l} = \varepsilon \sim \frac{v_0^3}{\ell_0}$$

  - But only contribution from "active" eddies to the energy E_ℓ

$$E_\ell = v_\ell^2\phi_\ell = v_\ell^2\left(\frac{\ell}{\ell_0}\right)^{3-D}$$

$$\Rightarrow \qquad v_\ell = v_0\left(\frac{\ell}{\ell_0}\right)^h \quad ; \quad h = 1/3-(3-D)/3$$

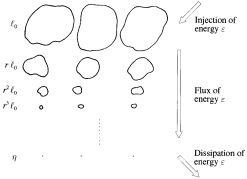
  - Two contributions for the estimation of the structure function $\langle\delta_r u^p\rangle$:

$$\langle\delta_r u^p\rangle \sim v_\ell^p\phi_\ell = v_0^p\left(\frac{\ell}{\ell_0}\right)^{ph}\left(\frac{\ell}{\ell_0}\right)^{3-D} = v_0^p\left(\frac{\ell}{\ell_0}\right)^{\zeta_p}$$

with $\zeta_p = ph-3-D = p/3+(3-D)(1-p/3)$ It is a linear function of p

Fractals and multifractals

The β model (continue)



▸ For $p=3$: $\zeta_3=1$ because it is based on the cascade picture (consistent with the 4/5 law)

▸ For $p=2$: $\zeta_2=2/3+(3-D)/3$

For the energy spectra with wavenumber $k=1/\ell$:

$$E(k)=\frac{\langle\delta_r u^2\rangle}{k}=\ell_0 v_0^2(k\ell_0)^{-5/3+(3-D)/3}$$

▸ D is the dimension of the intermittency cascade: The velocity fields has a **scaling exponent** h on a set of structures of Dimension D
⚠The dimension D need not be integer: **fractal dimension**
h and D are related $h=1/3+(3-D)/3$

- for $D=3$ we have the K41 theory
- Comparison with data for $p<8$ gives $D=2.8$; for larger p it fails

Fractals and multifractals

The bi-fractal model

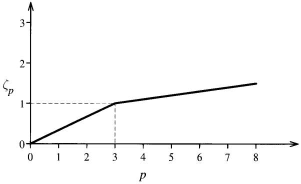
- Now consider 2 sets of structures with dimensions D_1 and D_2 and the associated scaling exponents h_1 and h_2:
- The structure functions is given by the superposition of 2 power laws:

$$\frac{\langle \delta_r u^p \rangle}{v_0^p} = \mu_1 \left(\frac{\ell}{\ell_0}\right)^{ph_1} \left(\frac{\ell}{\ell_0}\right)^{3-D_1} + \mu_2 \left(\frac{\ell}{\ell_0}\right)^{ph_2} \left(\frac{\ell}{\ell_0}\right)^{3-D_2}$$

- The power law with the **smallest** exponent dominates $(\ell/\ell_0 < 1)$

$$\frac{\langle \delta_r u^p \rangle}{v_0^p} \sim \left(\frac{\ell}{\ell_0}\right)^{\zeta_p} \quad ; \ \zeta_p = \min(ph_1 + 3 - D_1 \, , \, ph_2 + 3 - D_2)$$

Depending on p either the strucutres of set 1 or set 2 dominate.
Here by construction they cross for $p = 3$



Frisch 1995

Fractals and multifractals
The multi-fractal model[4]

▸ K41 assume a **global scale invariance** and obtain a scaling exponent $p/3$ for the structure functions

▸ Here consider a weakear form: "**local scale-invariance**"
In the multi-fractal formalism we consider and infinity of scaling exponent (vs 2 in the bi-fractal model)
A range of scaling exponent h is possible. For each h there is a fractal set with h dependent dimension $D(h)$ near which the velocity field scale with h.

$$\frac{\langle \delta_r u^p \rangle}{v_0^p} = \int_I d\mu(h) \left(\frac{\ell}{\ell_0} \right)^{ph+3-D(h)}$$

- the sum of the bi-fratal model is replace by an integral over the range of scaling exponent
- $d\mu$ is the measure that gives the weight of the various exponents (not relevant)
- the factor $(\ell/\ell_0)^{ph}$ is the contribution from $\delta_\ell v$
- the factor $(\ell/\ell_0)^{3-D(h)}$ is the proba. to be in the (fractal) set of dimension $D(h)$

---

[4] Parisi & Firsch (1985)

Fractals and multifractals

The multi-fractal model[5] (continue)

- for $\ell \ll \ell_0$ the power law with the smallest exponent dominates

$$\zeta_p = \inf_h \left[ph + 3 - D(h) \right]$$

(NB: Legendre transform between ζ_p and $D(h)$)

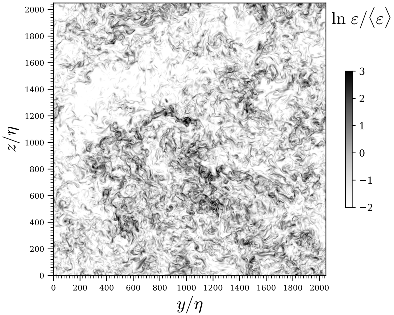
$$\Rightarrow \qquad \frac{\langle \delta_r u^p \rangle}{v_0^p} = \left(\frac{\ell}{\ell_0} \right)^{\zeta_p}$$

---

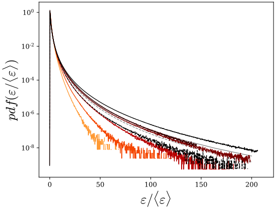
[5] Parisi & Firsch (1985))

Intermittency

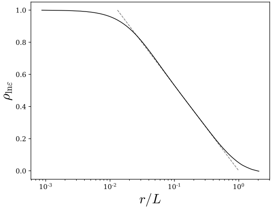
The dissipation rate presents very large fluctuations and is heterogeneous (famous Landau's remark[6])



PDF



autocorrelation



⇒ Long correlation, dependence of the forcing?

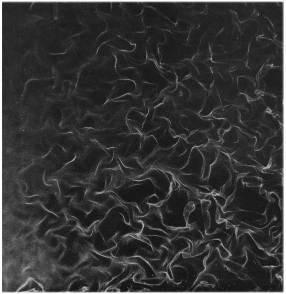
⇒ No Universality?

---

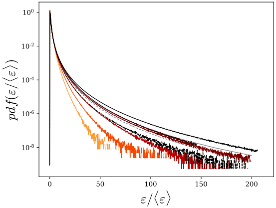
[6] Landau, L. D. and Lifshitz, E. M., Vol. 6, Fluid Mechanics (1944)

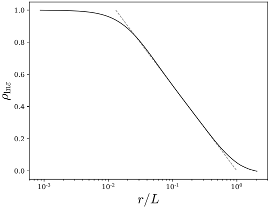
Intermittency

The dissipation rate presents very large fluctuations and is heterogeneous (famous Landau's remark[6])



M. Van Dyke, An Album of Fluid Motion


PDF


autocorrelation

⇒ Long correlation, dependence of the forcing?
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---

[6] Landau, L. D. and Lifshitz, E. M., Vol. 6, Fluid Mechanics (1944)

Refined Kolmogorov hypothesis[8]

$\langle \varepsilon \rangle$ is not the appropriate scale ⇒ Instead define a local scale for ε:

$$\varepsilon_\ell(\mathbf{x}) = \frac{3}{4\pi\ell^3} \int_{|h|<\ell} \varepsilon(\mathbf{x}+\mathbf{h}) d\mathbf{h}$$

(it is a coarse graining procedure)
⇒ Local velocity scales: $v_\ell = (\varepsilon_\ell \ell)^{1/3}$; $Re_\ell = v_\ell \ell / \nu$
⚠Energy transfer vs local average of ε: local imbalance, energy flux, positivity[7]

- Local similarity hypothesis: $\dfrac{\delta_\ell u}{v_\ell}$ presents a **universal distribution**: only dependent on Re_ℓ, not ℓ
- For the moments conditioned on the "local" value of ε_ℓ it gives:

$$\langle \delta_\ell u^p | \varepsilon_\ell \rangle = (\ell \varepsilon_\ell)^{p/3} \phi_p(Re_\ell)$$

- For $Re_\ell \gg 1$ assuming $\lim_{Re_\ell \to \infty} \phi_p(Re_\ell) = D_p$ (inertial range), we have:

$$\langle \delta_\ell u^p \rangle = D_p \ell^{p/3} \langle \varepsilon_\ell^{p/3} \rangle$$

What is the distribution of ε_ℓ ? Is it universal ? ⚠$\langle \varepsilon_\ell^{p/3} \rangle \neq \langle \varepsilon_\ell \rangle^{p/3}$

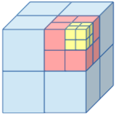
---

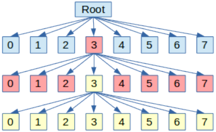
[7] *see Kraichnan JFM 1974 & Castaing et al. Physica D 1990*

[8] A. N. Kolmogorov JFM 1962 & Oboukhov JFM 1962

Multiplicative cascades

Energy cascade is naturally associated with multiplicative processes





Locally space-averaged dissipation over a volume of size $\ell = \ell_0 \lambda^n$:

$$\varepsilon_n = \varepsilon_0 \frac{\varepsilon_1}{\varepsilon_0} \dots \frac{\varepsilon_n}{\varepsilon_{n-1}} = \varepsilon_0 \prod_{i=1}^{n} \xi_i$$

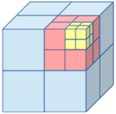
- Multiplication of many random numbers leads to very large fluctuations
- Scale invariance: → $\xi_i = \varepsilon_i / \varepsilon_{i-1}$ are independent and identically distributed random numbers ($\Rightarrow \xi_i > 0$)

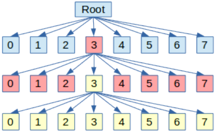
$$\begin{aligned}
\langle \varepsilon_n^p \rangle &= \langle \varepsilon_0^p \xi_1^p \xi_2^p \dots \xi_n^p \rangle \\
&= \varepsilon_0^p \langle \xi_1^p \rangle \langle \xi_2^p \rangle \dots \langle \xi_n^p \rangle \\
&= \varepsilon_0^p \langle \xi^p \rangle^n \\
&= \varepsilon_0^p \exp(n \ln \langle \xi^p \rangle) \\
&= \varepsilon_0^p \left(\frac{\ell}{\ell_0} \right)^{\alpha_p} \quad ; \quad \alpha_p = \ln \langle \xi^p \rangle / \ln \lambda
\end{aligned}$$

⚠ $\varepsilon_0 = \langle \varepsilon \rangle \Rightarrow \langle \xi_i \rangle = 1$
(but cascade may be non conservative if the 8 values of $\varepsilon_{\ell/2}$ are not related to ε_ℓ)

Multiplicative cascades

Black & white model[9]





In this simple model ξ_i can only take two values:

$$\xi_i = \begin{cases} 1/\beta & \text{with proba.} \quad \beta \\ 0 & \text{with proba.} \quad 1-\beta \end{cases}$$

- For the moments: $\langle \xi^p \rangle = \beta (1/\beta)^p = \beta^{1-p}$

- it gives for the moments of the dissipation: $\langle \varepsilon^p \rangle = \langle \varepsilon \rangle \left(\frac{\ell}{\ell_0} \right)^{\alpha_p}$

with $\alpha_p = \ln \langle \xi^p \rangle / \ln \lambda = (1-p) \ln \beta / \ln \lambda$

$$\langle \delta_\ell u^p \rangle \sim \ell^{p/3} \langle \varepsilon_\ell^{p/3} \rangle = v_0^p \left(\frac{\ell}{\ell_0} \right)^{p/3 + \alpha_{p/3}}$$

That gives the same behaviors as the β-model (with $\ln \beta / \ln \lambda = 3 - D$)

- Random cascade models present multifractal scaling.

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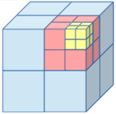
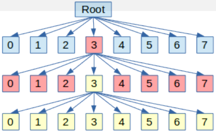
[9] *Novikov & Stewart (1964)*

Multiplicative cascade

The log-normal model





Returning to:

$$\varepsilon_n = \varepsilon_0 \prod_{i=1}^{n} \xi_i \rightarrow \ln \varepsilon_n / \varepsilon_0 = \sum_{i=1}^{n} \ln \xi_i$$

- from the the central limit theorem the distribution of $\ln \varepsilon_n / \varepsilon_0 = \sum_{i=1}^{n} \ln \xi_i$ tends to the **normal** distribution as $n \rightarrow \infty$
- $\Rightarrow$ ε_n presents a **log-normal distribution**

$$P_{\varepsilon_n}(x) = \mathscr{L}\mathscr{N}(x; \mu_n, \sigma_n^2) = \frac{1}{x} \frac{1}{\sqrt{2\pi\sigma_n^2}} \exp\left[-\frac{1}{2} \frac{(\ln x - \mu_n)^2}{\sigma_n^2} \right]$$

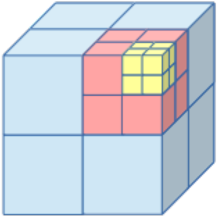
with parameters $\mu_n = n \langle \ln \xi \rangle$
and $\sigma_n^2 = n(\langle \ln^2 \xi \rangle - \langle \ln \xi \rangle^2)$

- Moments of a log-normal variables: $\langle x^p \rangle = \exp\left(p\mu_n + p^2 \sigma_n^2 / 2 \right)$
- This imposes $\mu_n = -\sigma_n^2/2$ to have $\langle \varepsilon_n \rangle = \langle \varepsilon_0 \rangle$
- with $n = \ln(\ell/\ell_0)/\ln \lambda$ we rewrite the parameter σ_n^2 as:

$$\sigma_n^2 = \underbrace{\frac{\langle \ln^2 \xi \rangle - \langle \ln \xi \rangle^2}{\ln \lambda}}_{9k} \ln(\ell/\ell_0) = 9k \ln(\ell/\ell_0)$$

Multiplicative cascade

The log-normal model (continue)

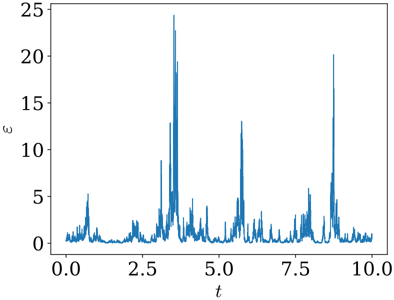


If we consider the limit $\ell \rightarrow \eta$.

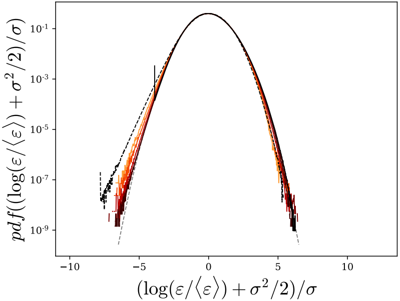
(⚠ $L/\eta = Re_L^{3/4} \sim Re_\lambda^{3/8}$)

We have the log-normal model for the local dissipation rate ε with parameters:
$\mu = -\sigma^2/2$
$\sigma^2 = A + B \ln Re_\lambda \approx 0.7 + 3/9 \ln Re_\lambda$

(KO 62 & Yeung et al. 2006)

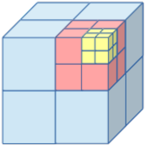


PDF of ε from DNS



Multiplicative cascade

The log-normal model (continue)



- from the moment of the log normal distribution:

$$\langle \varepsilon_\ell^p \rangle = \langle \varepsilon \rangle^p \exp(p(p-1)\sigma_n^2/2) = \langle \varepsilon \rangle^p \left(\frac{\ell}{\ell_0}\right)^{\alpha_p}$$

with $\alpha_p = \frac{9}{2}kp(p-1)$

The log-normality of ε implies for the structure function:

$$\begin{aligned} \langle \delta_\ell u^p \rangle &= D_p(\ell\langle\varepsilon\rangle)^{p/3}(\ell/L)^{\alpha_{p/3}} \\ &= v_0^p\,(\ell/L)^{p/3+\alpha_{p/3}} \end{aligned}$$

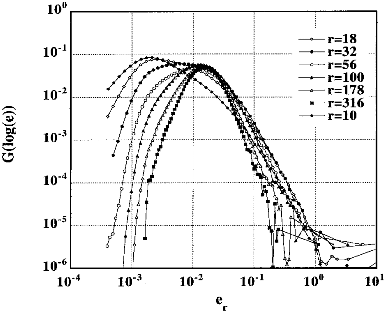
- $\alpha_{p/3} = 1/2kp(p-3)$ is call the **anomalous exponent** (not predicted by dimensional analysis)
- It characterizes the **absence of scale separation**:

→ Influence of the large scales on the small ones through the **intermittency**

The Kolmogorov theory

Intermittency: Experimental support

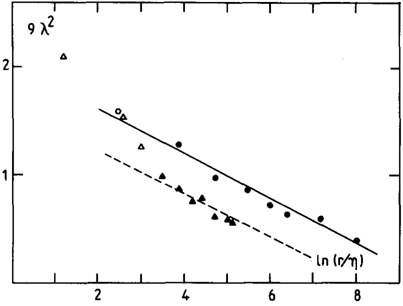
Approximate log-normality for ε_r
(But PDF of $\log \varepsilon$ are skewed[10])



Naert et al. Phys. D 1998

with scale dependent log-variance
$\sigma_r^2 = \langle (\log \varepsilon_r - \langle \log \varepsilon_r \rangle)^2 \rangle$

$$\sigma_r^2 = A + 9k \log L/r$$



Castaing et al., Physica D, 1990

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[10] *Vincent & Meneguzzi (1991)*

The Kolmogorov theory
Intermittency: Experimental support

Approximate log-normality for ε_r
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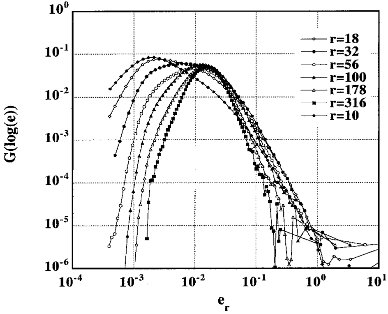
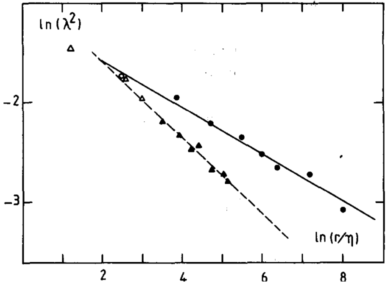


Naert et al. Phys. D 1998

with scale dependent log-variance
$\sigma_r^2 = \langle (\log \varepsilon_r - \langle \log \varepsilon_r \rangle)^2 \rangle$

or $\sigma_r^2 = (L/r)^{\beta_0/\log Re_\lambda}$?



Castaing et al., Physica D, 1990

Absence of similarity[11]?

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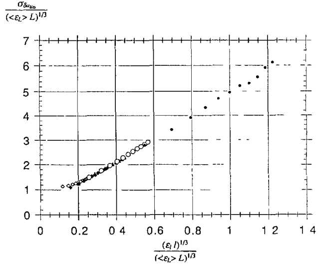
[10] *Vincent & Meneguzzi (1991)*

[11] *G. I. Barenblatt & N. Goldenfeld. PoF, 7, 1995. ; B. Castaing et al. Physica D 68, 1993.*

The Kolmogorov theory
Intermittency: Experimental support

conditional variance:
$\langle \delta_r u^2 | \varepsilon_r \rangle \sim (\varepsilon_r r)^{2/3}$

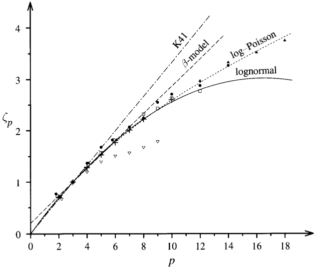


$Re_\lambda = 2700$,

$r/\eta = 24$, 60, 120, 250, 2500

Gagne et al., J. Phys., 1994

unconditional moments:
$\langle \delta_r u^p \rangle \sim r^{\zeta_p}$



Frisch 1995

The Kolmogorov theory

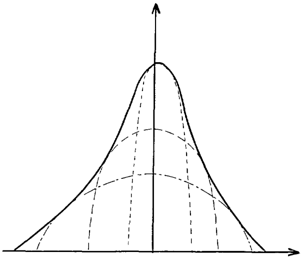
Intermittency: PDF of the velocity increments

Two levels of fluctuations:

1. fluctuations of $\delta_r u$ for a given transfer rate $\varepsilon_r \rightarrow P(\delta_r u|\varepsilon_r)$
2. fluctuations of the transfer rate itself $\rightarrow P(\varepsilon_r)$

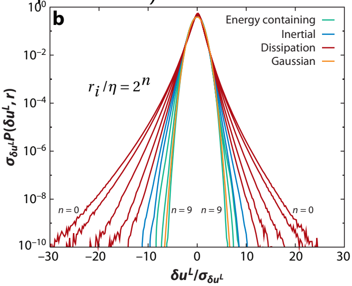
$$\Rightarrow P(\delta_r u) = \int_0^\infty P(\delta_r u|\varepsilon_r) P(\varepsilon_r) d\varepsilon_r$$

The statistics of the increments at scale r are equivalent to those at scale L within a random scaling factor (its distribution depends on r).



Gagne et al., J. Phys., 1994

$\Rightarrow$



Ishihara et al., Annu. Rev. 2009

The Kolmogorov theory

Beyond the log-normal model . . .

To be completed . . .
The problem of the moment
Non convexity of the intermittency exponent
Issues with the cascade model
Local vs Non-local (in scales) transfer
Multifractals models

Spectral representation

Fourier Transform

- What is a spectra ?
- Spectral representation of the velocity spectra
- Spectral decomposition of the energy budget (Lin equation)

Spectral decomposition of velocity field

For a **homogeneous** flow, the spectral description of the fields enables us to examine the properties of turbulence as a function of wavelength

We can apply the three-dimensional Fourier transform to the components of the velocity field u'_j :

$$\hat{u}'_j(\vec{k}) = \frac{1}{(2\pi)^3} \iiint u'_j(\vec{x}) e^{-\imath \vec{k}.\vec{x}} d^3 x$$

with $\vec{k}$ the wave vector, and $\imath = \sqrt{-1}$

and the inverse transformation:

$$u'_j(\vec{x}) = \iiint \hat{u}'_j(\vec{k}) e^{\imath \vec{k}.\vec{x}} d^3 k$$

⇒ Velocity fluctuations as the superposition of sinusoidal waves

Properties of the Fourier transform

A few properties of the Fourier transform are recalled:

- Linearity : $a f_1(\vec{x}) + b f_2(\vec{x}) \quad \rightarrow a \hat{f}_1(\vec{k}) + b \hat{f}_2(\vec{k})$
- prodct : $f_1(\vec{x}) f_2(\vec{x}) \quad \rightarrow \hat{f}_1(\vec{k}) * \hat{f}_2(\vec{k}) = \int_{-\infty}^{+\infty} \hat{f}_1(\vec{k} - \vec{k}') \hat{f}_2(\vec{k}) d^3 k'$
- derivative : $\dfrac{d f(\vec{x})}{d\vec{x}} \quad \rightarrow i \vec{k} \hat{f}(\vec{k})$
- Parseval's equality (conservation of energy)
  $\iiint_{-\infty}^{\infty} u_i u_i(\vec{x}) d^3 x = \iiint_{-\infty}^{\infty} \hat{u}_i(\vec{k}) \hat{u}_i^{\star}(\vec{k}) d^3 k$

Fourier transformation of the correlation tensor

The spectral tensor is given by the Fourier transform (3D) of the two-point correlation function: $\mathscr{R}_{ij}(\vec{r}) = \overline{u_i(\vec{x},t)\, u_j(\vec{x}+\vec{r},t)}$
(function only of r if turbulence is homogeneous):

$$\phi_{ij}(\vec{k}) = \frac{1}{(2\pi)^3} \iiint \mathscr{R}_{ij} e^{-\imath \vec{k}.\vec{r}} d^3 r$$

with $\vec{k}$ the wavevector, and $\imath = \sqrt{-1}$

and the inverse transformation :

$$\mathscr{R}_{ij} = \iiint \phi_{ij}(\vec{k}) e^{\imath \vec{k}.\vec{x}} d^3 k$$

$\mathscr{R}_{ij}$ tells us how velocities at points separated by a vector r are related. If we know these two point velocity correlations, we can deduce E(k). Hence the energy spectrum has the information content of the two-point correlation.

Fourier transformation of the correlation tensor

- We are essentially interested in the sum of the diagonal $\phi_{ij}(\vec{k})$:

$\phi_{ii} = \phi_{11} + \phi_{22} + \phi_{33}$ which represents the kinetic energy associated with a given wave vector.

$$\mathscr{R}_{ii}(0) = \overline{u'_i u'_i} = 3\,\mathscr{U}^2 = \iiint_{-\infty}^{\infty} \phi_{ii}(\vec{k})\,d^3k = \iiint_{-\infty}^{\infty} \overline{\hat{u}_i(\vec{k})\,\hat{u}_i^\star(\vec{k})}\,d^3k$$

$\frac{1}{2}\overline{\hat{u}_i(\vec{k})\,\hat{u}_i^\star(\vec{k})}\,d^3k = \frac{1}{2}\phi_{ii}(\vec{k})\,d^3k$ is the kinetic energy associated with wave vectors adjacent to $\vec{k}$ located in a volume d^3k of spectral space.

⚠Non isotropic flow...

Fourier transformation of the correlation tensor

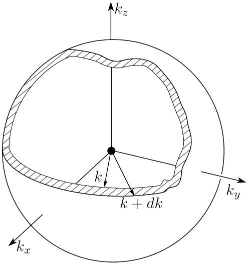
- $\phi_{ii}(\vec{k})$ accounts for orientation *(ϕ_{ii} can be different along on k_x, k_y or k_z)*.

Generally speaking, we want to know the energy at a given scale $k = |\vec{k}|$ without being interested in its direction.

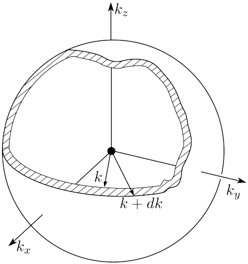
Directional information is eliminated by integrating on a sphere of radius k ($k^2 = k_i k_i$) to obtain the **three-dimensional energy spectrum** (or **spectral energy density**):

$$E(k) = \frac{1}{2} \oint \phi_{ii}(\vec{k}) d\sigma$$

($d\sigma$ surface element of the sphere of radius k)



Fourier transformation of the correlation tensor

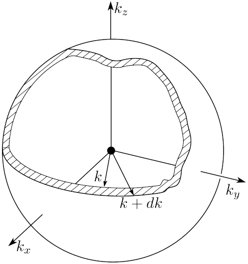


three-dimensional energy spectrum = **spectral energy density**:

$$E(k) = \frac{1}{2} \oint \phi_{ii}(\vec{k}) d\sigma$$

$E(k)dk$ represents the energy of modes with a wavenumber modulus between k et $k+dk$.

Fourier transformation of the correlation tensor



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$$E(k) = \frac{1}{2} \oint \phi_{ii}(\vec{k}) d\sigma$$

$E(k)dk$ represents the energy of modes with a wavenumber modulus between k et $k + dk$.

For isotropic flow, $\phi_{ii}(\vec{k})$ depends only on the modulus of the wave vector $k = |\vec{k}|$:

$$E(k) = 2\pi k^2 \phi_{ii}(\vec{k}) \qquad \textit{(noting that: } \oint d\sigma = 4\pi k^2 \textit{)}$$

Fourier transformation of the correlation tensor

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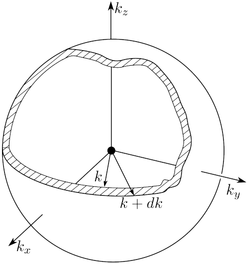

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(noting that: $\oint d\sigma = 4\pi k^2$)

Questions : What is the integral of $E(k)$?
What is the shape of $E(k)$ for white noise??

Spectral energy balance

We are interested in the equation for the evolution of $E(k)$ (isotropic energy spectrum at a given wavenumber). This equation includes terms describing the transfer from one scale to another via non-linear interactions.

To begin with, we apply the Fourier transform to the Navier-Stokes equation (their is some mathematical warning):

$$\frac{\partial u_i}{\partial t} - \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} = \underbrace{-\frac{1}{\rho}\frac{\partial P}{\partial x_i} - u_j \frac{\partial u_i}{\partial x_j}}_{\mathcal{N}\mathcal{L}_i}$$

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The left-hand member terms are linear:

- $\frac{\partial}{\partial t} u_i(\vec{x},t) \;\rightarrow\; \frac{\partial}{\partial t} \hat{u}_i(\vec{k},t)$
- $\nu \frac{\partial^2}{\partial x_j \partial x_j} u_i(\vec{x},t) \;\rightarrow\; -\nu k_j k_j \hat{u}_i(\vec{k},t)$

The terms on the right are non-linear: $\mathcal{N}\mathcal{L}_i(\vec{x},t) \;\rightarrow\; \widehat{\mathcal{N}\mathcal{L}_i}(\vec{k},t)$

Spectral energy balance

This gives the Navier-Stokes equation in spectral space:

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) \hat{u}_i(\vec{k}, t) = \widehat{\mathcal{N}\mathcal{L}_i}(\vec{k}, t)$$

Non-linear terms imply triple interactions between wave numbers.

To obtain the energy equation, multiply by $\hat{u}_i^\star$ and take the average:

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) \phi_{ii}(\vec{k}, t) = Re\left[\hat{u}_i^\star \widehat{\mathcal{N}\mathcal{L}_i}(\vec{k}, t)\right]$$

Spectral energy balance

With no advection and no pressure variation, the energy equation reduces to :

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right)\phi_{ii}(\vec{k},t) = 0$$

The energy density at a given wavenumber evolves independently of the energy content at other wavenumbers.
⇒ The wave numbers are not coupled.

The solution is : $\phi_{ii}(\vec{k},t) = \phi_{ii}(\vec{k},0)e^{-2\nu k^2 t}$.
The energy at wavenumber $\vec{k}$ decreases exponentially, with a rate that increases as the modulus of k increases.
⇒ Viscosity rapidly dampens movements on very small scales.

Spectral energy balance

Evolution of the isotropic velocity spectrum $E(k,t)$:

$$\frac{\partial}{\partial t}E(k,t) = T(k,t) - 2\nu k^2 E(k,t)$$

- $T(k,t)$ is obtained by integrating the contribution of the non-linear terms on the sphere of radius k : $T(k,t) = \frac{1}{2}\oint Re\left[\hat{u}_i^\star \widehat{\mathcal{N}\mathscr{L}_i}(\vec{k},t)\right] d\sigma$ → **Energy transfer between different wavenumbers due to nonlinear interactions**
- the viscous dissipation in an interval dk is equal to $2\nu k^2 E(k)dk$:

$$\langle\varepsilon\rangle = 2\nu\langle S_{ij}S_{ij}\rangle = \int_0^\infty 2\nu k^2 E(k,t)dk$$

Spectral energy balance

Evolution of the isotropic velocity spectrum $E(k,t)$:

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Let's look at the integral of the energy balance:

$$\frac{d}{dt}\int_0^\infty E(k,t)dk = \int_0^\infty T(k,t)dk - \int_0^\infty 2\nu k^2 E(k,t)dk$$

We find the energy balance $\frac{d\mathcal{K}}{dt} = -\varepsilon$ if : $\int_0^\infty T(k,t)dk = 0$.

⇒ Non-linear terms transfer energy between wave numbers, but do not change the overall energy.

Spectral energy balance

Let's add a forcing term to the energy equation in spectral space:

$$\frac{\partial}{\partial t}E(k,t) = F(k,t) + T(k,t) - 2\nu k^2 E(k,t)$$

$F(k,t)$ is the spectral forcing ($F(k,t) = \frac{1}{2}\oint Re\left[\hat{u}_i^\star \widehat{f}_i(\vec{k},t)\right] d\sigma$)

For stationary turbulence :

$$F(k,t) + T(k,t) = 2\nu k^2 E(k,t)$$

Integrating over k, we find that the rate of dissipation of kinetic energy is equal to the rate of energy injection: $\int_0^\infty F(k,t)dk = \langle\varepsilon\rangle$

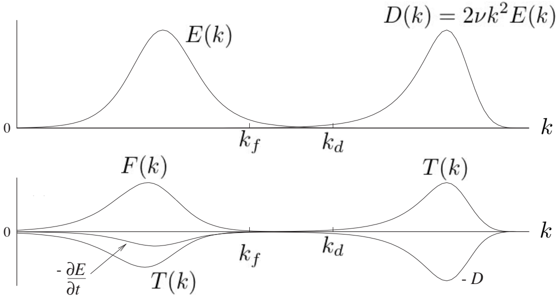
when the forcing is concentrated on a small spectral band centered around $k \approx k_f$, for $k > k_f$:

$$T(k,t) \approx 2\nu k^2 E(k,t)$$

Spectral energy balance

When viscosity is low enough that dissipation can be neglected at large scales, there is an intermediate range of wave numbers, between the forcing scale and dissipative scales, such as:

$$T(k,t) = 2\nu k^2 E(k,t) \approx 0$$



Spectral energy balance

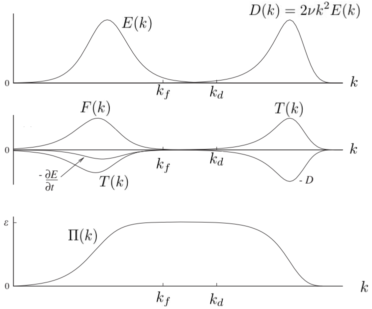
The kinetic energy flow through the "sphere" of radius k is defined by :

$$\Pi(k,t) = \int_k^\infty T(k',t)\,dk'$$

or

$$T(k,t) = -\frac{\partial \Pi(k,t)}{\partial k}$$

At intermediate scales, energy flow is constant: $\Pi(k) = \varepsilon$



- energy injection at a rate $\langle\varepsilon\rangle$ à $k < k_i$
- energy "cascades" to large wave numbers at a rate $\langle\varepsilon\rangle$
- energy is dissipated as heat at a rate $\langle\varepsilon\rangle$

Kolmogorov spectra

Kolmogorov's 1941 theory of the energy spectrum is based on the result that the ε dissipation rate controls the energy flow.
Energy transfer rate is independent of wavenumber at very high Reynolds numbers $(k_i \ll k_d)$. In the intermediate scale range, neither viscosity nor forcing is important: the energy flux k and the wavenumber k are the only parameters controlling the flow.

then the spectral energy density, are expressed:

$$E(k) = f(\varepsilon, k)$$

Dimensionally, we find that the possibility for f corresponds to the **Kolmogorov spectrum** :

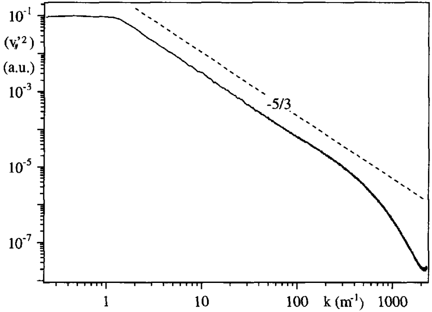
$$E(k) = C_K \, \varepsilon^{2/3} \, k^{-5/3}$$

$C_K \approx 1.5$ is a universal constant

Kolmogorov spectra

The Kolmogorov kinetic energy spectrum:

$$E(k) = C_K \varepsilon^{2/3} k^{-5/3}$$

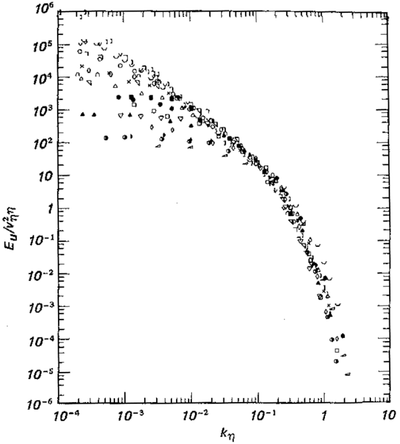


The range of spectral space for which energy follows the power law in "-5/3" is called the **inertial** range. In this range of scales, energy is transferred from larger to smaller scales.

Kolmogorov spectra

The Kolmogorov kinetic energy spectrum:

$$E(k) = C_K\,\varepsilon^{2/3}\,k^{-5/3}$$



Summary

Summary

- Many interacting scales
- High Reynolds number ⇒ local homogeneity/isotropy assumption.
- Three ranges: energy range, dissipative range and inertial range. But in actual turbulent flows, with non-homogenity at large scales, many other ranges, dominated by shear, rotation or buoyancy ...
- Energy transfer rate is the essential quantity for the modeling
- Scale separation (time of decay = time to traverse the cascade vs time scale of small enough eddies)

  ⇒ Independence between large scales and small scales (universality)?
  - K41: flow is scale-invariant in the inertial range
  - K62 + log-normal model: flow is not scale-invariant but the "scale-evolution" of the flow (the cascade process) is invariant ⇒ Intermittency

  Cascade & intermittency ⇒
- The small scales quantity, typically the **velocity derivatives**, present **violent** and **extreme events** and **Reynolds number dependence**
- They primarily depend on the local dissipation rate whose statistics depend on the Reynolds number through the intermittency of the cascade (depth of the cascade $\sim \ln Re_\lambda$)
- What about the **Lagrangian** description of the turbulence ? and the **acceleration** of fluid particles ?

Some useful relations

Large scales

Integral scales :
$$L = \int_0^\infty f(r)dr$$

Integral time :
$$T = L/U'$$

Averaged kinetic energy dissipation :
$$\langle \varepsilon \rangle \sim U'^3/L$$

Reynolds number (integral) :
$$Re = \frac{U'L}{\nu} \sim \left(\frac{L}{\eta}\right)^{4/3}$$

Inertial scales

Variance velocity increments:
$$\langle \delta_r u^2 \rangle \sim (\langle \varepsilon \rangle r)^{2/3}$$

Energy spectra:
$$E(k) \sim \langle \varepsilon \rangle^{2/3} k^{-5/3}$$

3rd order str fct:
$$\langle \delta_r u^3 \rangle = -\frac{4}{5} \langle \varepsilon \rangle r$$

Taylor microscale:
$$\lambda = \sqrt{15\nu/\langle \varepsilon \rangle} U'$$

Reynolds number (Taylor):
$$Re_\lambda = \frac{U'\lambda}{\nu} \sim Re^{1/2}$$

Dissipative scales

Kolmogorov length:
$$\eta = \nu^{3/4} \langle \varepsilon \rangle^{-1/4}$$

Kolmogorov time:
$$\tau_\eta = (\nu/\langle \varepsilon \rangle)^{1/2}$$

Kolmogorov velocity:
$$u_\eta = (\nu \langle \varepsilon \rangle)^{1/4}$$

Kolmogorov acceleration:
$$a_\eta = \langle \varepsilon \rangle^{3/4} \nu^{-1/4}$$

Reynolds number (dissipative):
$$Re_\eta = \frac{u_\eta \eta}{\nu} = 1$$

(*Left* to *right*) M. D. Millionshchikov, A. N. Kolmogorov, A. M. Yaglom, and R. Kraichnan at meeting at the Institut de Mécanique Statistique de la Turbulence, Marseille, 1961. (Photo courtesy J. L. Lumley.)