
Examination – Correction

Exercise 1 : Modelling of bioclogging in wastewater filters 20 points

A Local equations

- 0.5 pt 1. Flow incompressibility.
- 0.5 pt 2. A reactive term.
- 1 pt 3. It corresponds to flux conservation at the interface, as the two sides of the equation corresponds to the Fick law in each media, respectively.

B Generalities about volume averaging method

- 0.5 pt 1. A porous media fully filled with a liquid (no gas).
- 1 pt 2. The representative elementary volume is a volume range were porous media properties are about constant while moving the volume around its position, or while slightly changing its typical size.
- 0.5 pt 3. $\varepsilon_\gamma = V_\gamma/V$.
- 0.5 pt 4. $\langle C_\gamma \rangle^\gamma = \frac{1}{V_\gamma} \int_{V_\gamma} C_\gamma dV$.
- 0.5 pt 5. $\langle C_\gamma \rangle = \frac{1}{V} \int_{V_\gamma} C_\gamma dV$.
- 0.5 pt 6. $\langle C_\gamma \rangle = \varepsilon_\gamma \langle C_\gamma \rangle^\gamma$.
- 0.5 pt 7. $l_\gamma \ll r_0 \ll L$.
- 0.5 pt 8. $C_\gamma = \langle C_\gamma \rangle^\gamma + \widetilde{C}_\gamma$ and $C_\alpha = \langle C_\alpha \rangle^\alpha + \widetilde{C}_\alpha$. It is the decomposition in intrinsic average and spatial fluctuation.

C Volume averaging in the γ -phase

- 5 pts 1. We can permute the spatial average and the time derivative as the solid phase and the biofilm phase are considered as stationary.

The averaging theorem for the divergence applied to the first right-term of the local advection diffusion equation on C_γ gives (D_γ is uniform) :

$$\langle D_\gamma \text{div}(\overrightarrow{\text{grad}} C_\gamma) \rangle = D_\gamma \text{div}(\overrightarrow{\text{grad}} C_\gamma) + \frac{D_\gamma}{V} \int_{A_{\gamma\omega}} \vec{n}_{\omega\gamma} \cdot \overrightarrow{\text{grad}} C_\gamma dA. \quad (1)$$

Furthermore, using the other volume-averaging theorem we have :

$$\text{div}(\overrightarrow{\text{grad}} C_\gamma) = \text{div} \left[\overrightarrow{\text{grad}} \langle C_\gamma \rangle + \frac{1}{V} \int_{A_{\gamma\omega}} C_\gamma \vec{n}_{\gamma\omega} dA \right]. \quad (2)$$

For the second right-term we get, with the volume-averaging theorem on the divergence :

$$\langle \text{div}(\vec{v} C_\gamma) \rangle = \text{div} \langle \vec{v} C_\gamma \rangle + \frac{1}{V} \int_{A_{\gamma\omega}} C_\gamma \vec{v} \cdot \vec{n}_{\omega\gamma} dA. \quad (3)$$

Using the two relations provided, the assumption on upscaled incompressibility and the non-penetration at biofilm surface ($\vec{n}_{\omega\gamma} \cdot \vec{v} = 0$), we get :

$$\langle \text{div}(\vec{v} C_\gamma) \rangle = \langle \vec{v} \rangle \cdot \overrightarrow{\text{grad}} \langle C_\gamma \rangle. \quad (4)$$

By combining the former results we finally have :

$$\frac{\partial \langle C_\gamma \rangle}{\partial t} = D_\gamma \text{div} \left[\overrightarrow{\text{grad}} \langle C_\gamma \rangle + \frac{1}{V} \int_{A_{\gamma\omega}} C_\gamma \vec{n}_{\gamma\omega} dA \right] + D_\gamma \frac{1}{V} \int_{A_{\gamma\omega}} \vec{n}_{\omega\gamma} \cdot \overrightarrow{\text{grad}} C_\gamma dA - \langle \vec{v} \rangle \cdot \overrightarrow{\text{grad}} \langle C_\gamma \rangle. \quad (5)$$

- 1.5 pt 2. We remind that $\langle C_\gamma \rangle = \varepsilon_\gamma \langle C_\gamma \rangle^\gamma$. We assume homogeneity of the porous media (ε_γ is uniform). Thus,

$$\frac{\partial \langle C_\gamma \rangle^\gamma}{\partial t} = D_\gamma \text{div} \left[\overrightarrow{\text{grad}} \langle C_\gamma \rangle^\gamma + \frac{1}{V_\gamma} \int_{A_{\gamma\omega}} C_\gamma \vec{n}_{\gamma\omega} dA \right] + \frac{D_\gamma}{V_\gamma} \int_{A_{\gamma\omega}} \vec{n}_{\omega\gamma} \cdot \overrightarrow{\text{grad}} C_\gamma dA - \langle \vec{v} \rangle \cdot \overrightarrow{\text{grad}} \langle C_\gamma \rangle^\gamma. \quad (6)$$

D Volume averaging in the ω -phase

- 0.5 pt 1. $\langle R_\omega \rangle^\omega = -\alpha \mu_{max}$.
- 0.5 pt 2. $\langle R_\omega \rangle^\omega = -\alpha \mu_{max} \frac{\langle C_\omega \rangle^\omega}{K}$.
- 1.5 pt 3. We replace $\langle R_\omega \rangle^\omega$ by its expression given in the wording. We replace the the term under the divergence by the closure equation. Finally, the boundary condition of no C_ω gradient at $\sigma - \omega$ interface leads to

$$\frac{\partial \langle C_\omega \rangle^\omega}{\partial t} = \text{div} \left[\overline{\overline{D_\omega}} \cdot \overrightarrow{\text{grad}} \langle C_\omega \rangle^\omega \right] - \alpha \mu_{max} \frac{\langle C_\omega \rangle^\omega}{\langle C_\omega \rangle^\omega + K}. \quad (7)$$

E One-equation model

The two volume-averaged equations for fluid and biofilm phases are respectively :

$$\frac{\partial \langle C_\gamma \rangle^\gamma}{\partial t} = \text{div} \left[\overline{\overline{D_\gamma}} \cdot \overrightarrow{\text{grad}} \langle C_\gamma \rangle^\gamma \right] + \frac{D_\gamma}{V} \int_{A_{\gamma\omega}} \vec{n}_{\gamma\omega} \cdot \overrightarrow{\text{grad}} C_\gamma dA - \langle \vec{v} \rangle \cdot \overrightarrow{\text{grad}} \langle C_\gamma \rangle^\gamma, \quad (8)$$

$$\frac{\partial \langle C_\omega \rangle^\omega}{\partial t} = \text{div} \left[\overline{\overline{D_\omega}} \cdot \overrightarrow{\text{grad}} \langle C_\omega \rangle^\omega \right] - \alpha \mu_{max} \frac{\langle C_\omega \rangle^\omega}{\langle C_\omega \rangle^\omega + K}. \quad (9)$$

- 0.5 pt 1. These are the spatial fluctuations which depart from the averaged concentrations.
- 1 pt 2. This is similar to the local thermal equilibrium assumption in thermal transfer in porous media.
- 3 pts 3. By adding equation 8 multiplied by ε_γ to equation 9 multiplied by ε_ω and using the thermodynamic equilibrium and the fact that averaged concentrations are much larger than the fluctuations, we get :

$$\frac{\partial \langle C \rangle}{\partial t} = \text{div} \left[\left(\varepsilon_\gamma \overline{D_\gamma} + \varepsilon_\omega \overline{D_\omega} \right) \overrightarrow{\text{grad}} \langle C_\gamma \rangle^\gamma \right] + (D_\omega \overrightarrow{\text{grad}} \varepsilon_\omega - \varepsilon_\gamma \langle \overrightarrow{v} \rangle) \cdot \overrightarrow{\text{grad}} \langle C_\gamma \rangle^\gamma - \alpha \mu_{max} \frac{\langle C_\omega \rangle^\omega}{\langle C_\omega \rangle^\omega + K}. \quad (10)$$

For the integral present in equation 8, note that $C_\gamma \approx \langle C_\gamma \rangle^\gamma$ so the gradient can be removed from the integral, and the remaining integral is replaced by the porosity gradient formula $\overrightarrow{\text{grad}} \varepsilon_\omega = \frac{1}{V} \int_{A_{\gamma\omega}} \overrightarrow{n}_{\gamma\omega} dA$ provided in the lecture.

Exercise 2 : Article analysis

20 points

We propose to study an article entitled “Inertia onset in disordered porous media flow” written by Damian Śniezek *et al.*. Before beginning the questions, have a quick overview of the article to remind its structure and the main topic addressed. **You should have already read the article.** Your answers should be written with your own words, do not paraphrase.

Questions

A Generalities about the article

- 0.25 pt 1. 2024.
- 0.25 pt 2. Physical Review E
- 2 pts 3. The authors aims to study precisely the transition from Darcy to inertial regime in flows through porous media. The use Direct Numerical Simulations and explore a range of Reynolds number between 0.01 and 100. They propose to go further than the friction factor to detect inertia onset by studying three other parameters : hydraulic tortuosity, kinetic energy localization and negative streamwise velocities. With these quantities the expect to demonstrate the decrease of preferential paths when the inertia effect occurs.

B Introduction

- 0.5 pt 1. The geometric tortuosity for a system of macroscopic length L and effective length L' is defined as $\tau = L'/L > 1$.
- 1 pt 2. Yes it can as the quantity transported depend on the transport mechanism. For example, hydraulic paths do not follow the shorter geometric paths in a porous media (they are smoother).
- 1 pt 3. Here \mathcal{K} is in m^3 whereas in the lecture K is in m^2 .
- 0.5 pt 4. $\beta_l = \beta \mathcal{K} / \rho$ by identification of the Darcy-Forcheimer relation provided in the article and in the lecture.
- 0.5 pt 5. $Re_p = \frac{\rho \langle v \rangle \ell_\alpha}{\eta \varepsilon}$.
- 0.5 pt 6. According to the literature, inertia onset is between $Re = 1$ and $Re = 10$, which is consistent with the lecture.
- 0.5 pt 7. “A new look at porous media fluid mechanics—Darcy to turbulent” (reference [18]).

C Material and method

- 0.5 pt 1. $\varphi = \frac{V_\alpha}{V_\alpha + V_\sigma}$ is the porosity, where V_α is the volume of the fluid phase and V_σ the one of the solid phase.
- 1.5 pt 2. We can base on the slide 12 of the *Upscaling to porous media* course. We have one microscopic, local relationship between a forcing term, a response term and the local media properties. Here it is a Stokes equation coupled to incompressibility. We take volume average on the Representative Elementary Volume (REV), i.e. a volume whose porous structure and averaged physical quantity are uniform. With the appropriate gradient and divergence theorems, we obtain an average of the Stokes and continuity equations, with interfacial integrals which need to be estimated. To do that, we split the control and response variables as a sum of their average on the REV and their fluctuations with respect to this average. We can then obtain a second equation on the fluctuations. Hypotheses of homogeneity of the porous media and separation of the scales (pore, REV, macro) allow to simplify the equations. Then we use a closure relation linking linearly the fluctuations of a field to its average in order to close the system and obtain an equation relating only the means of the forcing and response terms. In the case of fluid flow, a Brinkman correction remains. Using the scale separation, we can neglect it and we get the Darcy's law. A similar process can be used for Darcy-Forchheimer law.
- 0.5 pt 3. For a general porous media we get $\text{div}\langle \vec{u} \rangle^\alpha = -\varphi^{-1} \overrightarrow{\text{grad}}\varphi \cdot \langle \vec{u} \rangle$.
- 1 pt 4. u_{in} is the average inlet velocity. As the flow rate is conserved, the velocity averaged on the cross-section of the device is too. Consequently u_{in} correspond to the velocity averaged over the whole porous media, not only the fluid phase. It is the superficial average velocity.
- 0.25 pt 5. OpenFOAM.

D Results and Discussion

- 1 pt 1. It is not the case here as a dimension analysis of the Darcy-Forchheimer law provided by the authors in the first column allows to get that $[\beta] = \text{m}^{-4} \cdot \text{s}$. This does not allow to have f without dimension. There is an error in the article, either in the definition of β or in the definition of f .
- 0.5 pt 2. The dimension of β_l is m^{-1} .
- 0.5 pt 3. It can be easily demonstrated that yes, it is.
- 1.5 pt 4. We can write the Darcy-Forchheimer law with order of magnitude :

$$u_{in} = -\frac{\mathcal{K}}{\mu L} \frac{\Delta p}{L} - \beta \mathcal{K} u_{in}^2. \quad (11)$$

Using the definition of Re and f we get after some straightforward computations :

$$f = \frac{L^2 \mu}{\rho D} + \frac{L^2}{\beta \mathcal{K}} \frac{1}{Re}. \quad (12)$$

- 1 pt 5. At low Reynolds, the second term dominates, we have $f \propto Re^{-1}$ which is what we observe on the graph. For larger Reynolds numbers we get a transition towards a constant regime (first term).
- 0.5 pt 6. In a straight pipe the velocity is only streamwise so $\langle u_l \rangle = \langle |\vec{u}| \rangle \Rightarrow \tau = 1$.
- 0.5 pt 7. These are the standard deviation of eight different simulations.
- 0.5 pt 8. At low Re , viscosity dominates and flow streamlines follow the porous structure which results in constant tortuosity with Re . For higher Re , the flow does not follow the structure and adopts a more direct path leading to decreasing tortuosity.
- 0.5 pt 9. Figure 3 represents the proportion of negative streamwise velocity in the pore volume.

- 0.5 pt 10. Authors reveal that the appearance of backward flows appear much latter (in terms of Reynolds number) than the decrease of tortuosity (figure 2). Yet, these two phenomena are related to inertia onset.
- 0.5 pt 11. Channeling effect is a phenomenon where flow has preferential path : most of the mass transport is restricted to some specific path through the porous media. Their volume fraction are rather small.
- 0.25 pt 12. Inertial forces become higher than viscous forces letting the fluid to spread out through the porous media.

E Conclusions

- 1 pt 1. The authors observed that the inertia onset is one order of magnitude lower (in terms of Re) than expected by the literature. They demonstrated that by measuring the tortuosity of the flow and kinetic energy spatial repartition.
- 0.5 pt 2. The authors observed a weakening of flow channeling when Re increases, with more dispersed flow.