Département MFEE
$3^{ème}$ année Parcours FEP/M2 FEIP
Toulouse INP – ENSEEIHT

Transfers in Porous Media
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Examination

Instructions and recommendations

1. **You may read these instructions, but do not turn the page or begin the work until instructed.**
2. Lecture notes provided at the end of the class are allowed during the examination. You can also have your personal notes. Any electronic device is forbidden, including calculator.
3. Communication between students is not allowed.
4. **Please**, add a 10-square margin at the top of your first page, and a 4-square margin at the left or right of each page.
5. Answers can be given in English, French or Spanish. Language level will not be a notation criterion.
6. Time limit : 120 minutes. Please stop working when asked.
7. Detail all work and assumptions in order to get the maximal score. A clear and detailed redaction is required.
8. This examination contains two sections. The first one is an exercise. The second one is a guided analysis of an article. Score is balanced between these two parts.
9. This exam is *probably* too long for two hours, so do not worry if you do not finish it.
10. Consequently, you should split equitably your time between both parts of the exam.
11. You have already a printed version of the article.
12. The present document contains 7 pages.

Exercise 1 : Modelling of bioclogging in wastewater filters 20 points

This exercise aims at modelling the bioclogging due to biofilm development in a porous media (e.g. soil, filter or bed reactor). Bioclogging is the process of reduction of hydrodynamic permeability of a porous media due to deposition and devlopment of biomass at its surface. A biofilm is a colony of bacteria adhered at a surface and wrapped in an extracellular polymeric matrix produced by the bacteria.

We consider a **homogeneous** saturated porous media. A suspension of bacteria flow through the porous media. Some bacteria adhere to the solid surface and develop biofilm. Nutrients are present in the fluid and are consumed by the bacteria that use them to proliferate and develop more biofilm. We assume that the biofilm volume fraction is homogeneous in the porous media.

The solid (or filter) media is named σ-phase. The fluid is named γ-phase. The biofilm is named ω-phase (see figure 1 (II)). We will use the volume averaging method to determine the large (or field) scale equations for biofilm growth and nutrient concentration in the porous media.

We consider a Cartesian coordinate system (x, y, z). A Representative Elementary Volume (REV) is assumed to exist at Darcy-scale (see Figure 1 (II)). We assume that scale separation is valid.

C_γ is the nutrient concentration in the fluid and C_ω is the nutrient concentration in the biofilm. The fluid cannot pass through the $\gamma - \omega$ interface, that means there is no fluid in the biofilm.

Notations :

- t : time ;
- $\overrightarrow{v}(x, y, z, t)$: local velocity vector ;
- $C_\gamma(x, y, z, t)$: local nutrient concentration in the fluid phase ;
- $C_\omega(x, y, z, t)$: local nutrient concentration in the biofilm ;
- $\overrightarrow{n}_{\gamma\omega}$: normal vector at the $\gamma - \omega$ interface (oriented towards ω) ;
- $\overrightarrow{n}_{\omega\gamma}(= -\overrightarrow{n}_{\gamma\omega})$: normal vector at the $\omega - \gamma$ interface (oriented towards γ) ;
- $\overrightarrow{n}_{\sigma\omega}$: normal vector at the $\sigma - \omega$ interface (oriented towards ω) ;
- $\overrightarrow{n}_{\omega\sigma}(= -\overrightarrow{n}_{\sigma\omega})$: normal vector at the $\omega - \sigma$ interface (oriented towards σ) ;
- $\overrightarrow{n}_{\gamma\sigma}$: normal vector at the $\gamma - \sigma$ interface (oriented towards σ) ;
- $\overrightarrow{n}_{\sigma\gamma}(= -\overrightarrow{n}_{\gamma\sigma})$: normal vector at the $\sigma - \gamma$ interface (oriented towards γ) ;
- $A_{\gamma\omega}$: interfacial area of the $\gamma - \omega$ system ;
- $A_{\omega\sigma}$: interfacial area of the $\omega - \sigma$ system ;
- r_0 : typical scale of the REV ;
- ℓ_γ : typical pore scale (only fluid, no biofilm, see Figure 1 (I)) ;
- ℓ_ω : typical biofilm thickness ;
- L : typical scale of the field scale ;
- V : typical volume of the REV ;
- V_γ : typical volume of the γ-phase in V ;
- V_σ : typical volume of the σ-phase in V ;
- V_ω : typical volume of the ω-phase in V ;
- ε_γ : fluid volume fraction (commonly named porosity) ;
- ε_ω : biofilm volume fraction ;
- ε_σ : solid volume fraction ;
- D_γ : scalar that represents the diffusion coefficient of nutrients in the fluid ;
- D_ω : scalar that represents the diffusion coefficient of nutrients in the biofilm ;

In order to develop the mathematical expressions describing the transport process, a certain of necessary assumptions must be made : (i) σ-phase is assumed to be impermeable to all chemical species and absence of biological reaction in this phase (ii) Fick's law governs diffusion in the γ-phase and ω-phase ; (iii) ω-phase is permeable to all chemical species ; (iv) the biofilm is assumed to steady-state form thus ε_ω is constant with time ; (v) all the solid phase is covered by biofilm.

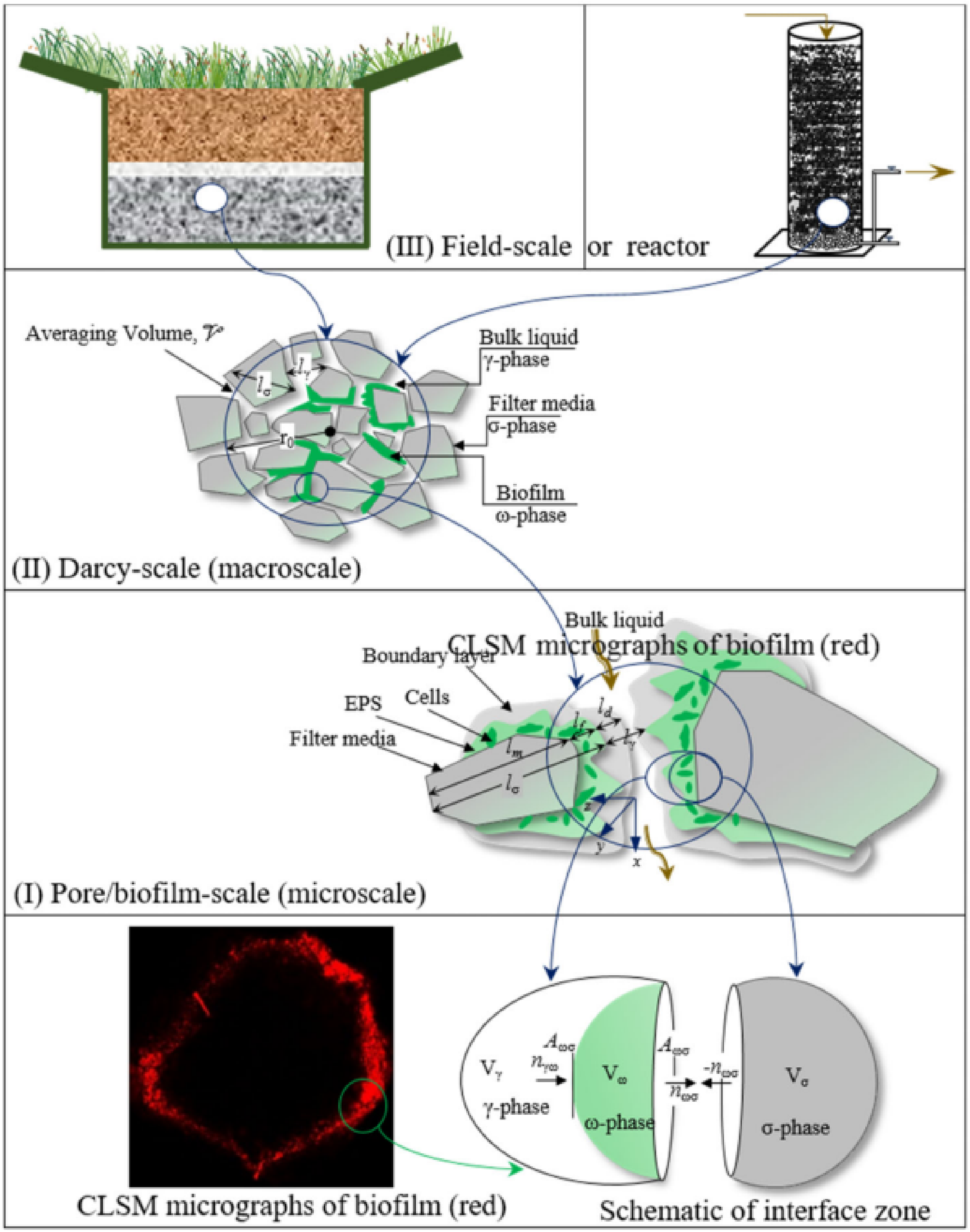

FIGURE 1 – Sketch of the different scales of the problem.

A Local equations

We assume that

$$\operatorname{div} \vec{v} = 0. \tag{1}$$

Under the previous assumptions, we consider that the local nutrient concentration in the γ phase follows the advection-diffusion equation [1]

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[1] Δ represents the Laplacian operator.

$$\frac{\partial C_\gamma}{\partial t} = D_\gamma \mathrm{div}(\overrightarrow{\mathrm{grad}}(C_\gamma)) - \mathrm{div}(\overrightarrow{v} C_\gamma), \tag{2}$$

where we assumed that D_γ is a constant scalar.

In the ω phase, the local nutrient concentration follows the reaction-diffusion equation :

$$\frac{\partial C_\omega}{\partial t} = D_\omega \mathrm{div}(\overrightarrow{\mathrm{grad}}(C_\omega)) + R_\omega, \tag{3}$$

where we assumed that D_ω is a constant scalar.

The **boundary conditions** are the following :

$$\begin{align}
\text{At} \quad & A_{\gamma\omega}, \quad C_\gamma = C_\omega, \tag{4}\\
\text{At} \quad & A_{\gamma\omega}, \quad -D_\gamma \overrightarrow{n}_{\gamma\omega} \cdot \overrightarrow{\mathrm{grad}} C_\gamma = -D_\omega \overrightarrow{n}_{\gamma\omega} \cdot \overrightarrow{\mathrm{grad}} C_\omega, \tag{5}\\
\text{At} \quad & A_{\gamma\omega}, \quad \overrightarrow{n}_{\gamma\omega} \cdot \overrightarrow{v} = 0, \tag{6}\\
\text{At} \quad & A_{\omega\sigma}, \quad -\overrightarrow{n}_{\omega\sigma} \cdot \overrightarrow{\mathrm{grad}} C_\omega = 0. \tag{7}
\end{align}$$

0.5 pt 1. What is the underlying hypothesis of equation 1 ?

0.5 pt 2. What does R_ω represents in eq. 3 ?

1 pt 3. What is the meaning of the boundary condition written eq. 5 ?

B Generalities about volume averaging method

0.5 pt 1. What is a the definition of a saturated porous media ?

1 pt 2. Give a definition for the Representative Elementary Volume (REV).

0.5 pt 3. What is the relation between ε_γ, V and V_γ ?

0.5 pt 4. Using the same notations as in the lectures, give the definition of the intrinsic average of the nutrient concentration in the γ phase.

0.5 pt 5. Using the same notations as in the lectures, give the definition of the superficial average of the nutrient concentration in the γ phase.

0.5 pt 6. What is the relation between the two previous quantities ?

0.5 pt 7. How does the scale separation translate in terms of r_0, ℓ_γ and L ?

0.5 pt 8. Using the same notations as in the lecture, propose a decomposition for both C_γ and C_ω, involving average and fluctuation.

We remind the two main spatial averaging theorems for a scalar Ψ or a vector $\overleftarrow{\Psi}$.

$$\begin{align}
\langle \overrightarrow{\mathrm{grad}} \Psi \rangle &= \overrightarrow{\mathrm{grad}} \langle \Psi \rangle + \frac{1}{V} \int_{A_{xy}} \Psi \overrightarrow{n_{xy}} dS, \tag{8}\\
\langle \mathrm{div} \overrightarrow{\Psi} \rangle &= \mathrm{div} \langle \overrightarrow{\Psi} \rangle + \frac{1}{V} \int_{A_{xy}} \overrightarrow{\Psi} \cdot \overrightarrow{n_{xy}} dS. \tag{9}
\end{align}$$

A_{xy} represents the interface area between two phases x and y and $\overrightarrow{n}_{xy}$ the normal vector at the interace, oriented toward the y-phase.

C Volume averaging in the γ-phase

We assume that $\mathrm{div}\langle\vec{v}\rangle = 0$ and that $\langle\vec{v}C_\gamma\rangle = \langle\vec{v}\rangle\langle C_\gamma\rangle$.

We remind the vector analysis formula $\mathrm{div}(f\vec{g}) = f\mathrm{div}(\vec{g}) + \vec{g}\cdot\overrightarrow{\mathrm{grad}}f$.

5 pts 1. Apply the volume averaging theorem to the local equation 2 on C_γ. Using boundary conditions and the previous relations and assumptions, show that

$$\frac{\partial\langle C_\gamma\rangle}{\partial t} = D_\gamma\mathrm{div}\left[\overrightarrow{\mathrm{grad}}\langle C_\gamma\rangle + \frac{1}{V}\int_{A_{\gamma\omega}} C_\gamma\vec{n}_{\gamma\omega}dA\right] + \frac{D_\gamma}{V}\int_{A_{\gamma\omega}}\vec{n}_{\omega\gamma}\cdot\overrightarrow{\mathrm{grad}}C_\gamma dA - \langle\vec{v}\rangle\cdot\overrightarrow{\mathrm{grad}}\langle C_\gamma\rangle. \tag{10}$$

1.5 pt 2. From this equation, give the equation on the intrinsic average $\langle C_\gamma\rangle^\gamma$ and the superficial average $\langle\vec{v}\rangle$ which does not imply ε_γ.

D Volume averaging in the ω-phase

With a similar procedure, we can get the volume-averaged equation in the ω-phase :

$$\frac{\partial\langle C_\omega\rangle^\omega}{\partial t} = D_\omega\mathrm{div}\left[\overrightarrow{\mathrm{grad}}\langle C_\omega\rangle^\omega + \frac{1}{V_\omega}\int_{A_{\omega\sigma}} C_\omega\vec{n}_{\omega\sigma}dA\right] + \frac{D_\omega}{V_\omega}\int_{A_{\omega\sigma}}\vec{n}_{\omega\sigma}\cdot\overrightarrow{\mathrm{grad}}C_\omega dA + \langle R_\omega\rangle^\omega. \tag{11}$$

We consider that the reaction term follows a Monod-like behaviour :

$$R_\omega = -\alpha\mu_{max}\frac{C_\omega}{C_\omega + K}, \tag{12}$$

where α, μ_{max} and K are constant.

0.5 pt 1. In the case where $K \ll C_\omega$, what is the expression of $\langle R_\omega\rangle^\omega$?

0.5 pt 2. In the case where $K \gg C_\omega$, compute the expression of $\langle R_\omega\rangle^\omega$.

1.5 pt 3. We can demonstrate that more generally $\langle R_\omega\rangle^\omega = -\alpha\mu_{max}\dfrac{\langle C_\omega\rangle^\omega}{\langle C_\omega\rangle^\omega + K}$.

Furthermore, we can define as closure relation a diffusivity tensor as

$$\overline{\overline{D_\omega}}\cdot\overrightarrow{\mathrm{grad}}\langle C_\omega\rangle^\omega = D_\omega\overrightarrow{\mathrm{grad}}\langle C_\omega\rangle^\omega + \frac{D_\omega}{V_\omega}\int_{A_{\omega\sigma}}\vec{n}_{\omega\sigma}C_\omega dA. \tag{13}$$

With these two assumptions and using a boundary condition, simplify eq. 11.

E One-equation model

We can also define a diffusivity tensor for the γ-phase such as :

$$\overline{\overline{D_\gamma}}\cdot\overrightarrow{\mathrm{grad}}\langle C_\gamma\rangle^\gamma = D_\gamma\overrightarrow{\mathrm{grad}}\langle C_\gamma\rangle^\gamma + \frac{D_\gamma}{V_\gamma}\int_{A_{\gamma\omega}}\vec{n}_{\gamma\omega}C_\gamma dA. \tag{14}$$

The two volume-averaged equations for fluid and biofilm phases are then respectively :

$$\frac{\partial \langle C_\gamma \rangle^\gamma}{\partial t} = \text{div}\left[\overline{\overline{D_\gamma}} \cdot \overrightarrow{\text{grad}} \langle C_\gamma \rangle^\gamma\right] + \frac{D_\gamma}{V_\gamma} \int_{A_{\gamma\omega}} \overrightarrow{n}_{\gamma\omega} \cdot \overrightarrow{\text{grad}} C_\gamma dA - \langle \overrightarrow{v} \rangle \cdot \overrightarrow{\text{grad}} \langle C_\gamma \rangle^\gamma, \tag{15}$$

$$\frac{\partial \langle C_\omega \rangle^\omega}{\partial t} = \text{div}\left[\overline{\overline{D_\omega}} \cdot \overrightarrow{\text{grad}} \langle C_\omega \rangle^\omega\right] - \alpha \mu_{max} \frac{\langle C_\omega \rangle^\omega}{\langle C_\omega \rangle^\omega + K}. \tag{16}$$

We can decompose the concentrations as :

$$C_\gamma = \langle C_\gamma \rangle^\gamma + \widetilde{C_\gamma}, \tag{17}$$

$$C_\omega = \langle C_\omega \rangle^\omega + \widetilde{C_\omega}. \tag{18}$$

We can define a weighted average of concentration :

$$\langle C \rangle = \varepsilon_\gamma \langle C_\gamma \rangle^\gamma + \varepsilon_\omega \langle C_\omega \rangle^\omega. \tag{19}$$

0.5 pt 1. What do represent $\widetilde{C_\omega}$ and $\widetilde{C_\gamma}$?

1 pt 2. We assume that we are at thermodynamic equilibrium so that concentration is homogeneous in γ and ω phase, $C_\omega = C_\gamma$. To which similar assumption made during the lecture this assumption can be linked ?

3 pts 3. We consider that $\langle C_\gamma \rangle^\gamma \gg \widetilde{C_\gamma}$ and $\langle C_\omega \rangle^\omega \gg \widetilde{C_\omega}$. Deduce from the previous results and assumption an one-equation model for the temporal variation of $\langle C \rangle$.- You will also need a mathematical relation between $\overrightarrow{\text{grad}}\varepsilon_\gamma$ and an integral detailed in the lecture.

Exercise 2 : Article analysis 20 points

We propose to study an article entitled "Inertia onset in disordered porous media flow" written by Damian Śnieżek *et al.*. Before beginning the questions, have a quick overview of the article to remind its structure and the main topic addressed. **You should have already red the article.** Your answers should be written with your own words, do not paraphrase.

Questions

A Generalities about the article

0.25 pt 1. Which is the year of publication of this article ?

0.25 pt 2. In which journal has it been published ?

2 pts 3. In few lines and without paraphrasing the text, summarize the objectives of the authors.

B Introduction

0.5 pt 1. Give the definition of the geometric tortuosity.

1 pt 2. Can the tortuosity depend on the considered transport mechanism ? Propose an example.

1 pt 3. What is the dimension of the permeability $\mathcal{K}$? How is it related to the permeability K defined during the lecture ?

0.5 pt 4. Using the expression of the Darcy-Forchheimer law proposed in the lecture, give the relation between the parameter β of the article with the lecture's one. You can note β_l the parameter β defined in the lecture.

0.5 pt 5. Give a definition of the Reynolds number cited in the article with the lecture's notations.

0.5 pt 6. For which range of Reynolds number do the inertia effects appear according to the literature ? Is it consistent with what we discussed during the lectures ?

0.5 pt 7. Give the title of the article which revealed for the first time experimental evidences of transition between Darcy and inertial regimes in 2D porous media.

C Material and method

0.5 pt 1. What is the definition of φ ?

1.5 pt 2. Explain in few lines the process we generally used to get macroscopic relations such as Darcy or Darcy-Forchheimer laws from local equations (1) and (2).

0.5 pt 3. What is the volume averaged equation obtained from equation (1) in the most general case ?

1 pt 4. Is the inlet velocity u_{in} a superficial or an intrinsic velocity ? Justify.

0.25 pt 5. Which software do the authors use to perform their simulations ?

D Results and Discussion

1 pt 1. The friction parameter f is expected to be a dimensionless number. Is it the case with the definition of β proposed in the article ? What do you conclude ?

0.5 pt 2. What is the dimension of β_l, the parameter β provided in the lecture ?

0.5 pt 3. If we use β_l instead of β in the definition of f, is f dimensionless ?

1.5 pt 4. Using the Darcy-Forchheimer law and the definitions of f and Re provided in the article, show that :

$$f = \frac{L^2\mu}{\rho D} + \frac{L^2}{\beta\mathcal{K}}\frac{1}{Re}. \tag{20}$$

1 pt 5. Explain the behaviour of the graph in figure 1 in the light of the previous relation. Pay attention to the fact that the graph is plot in log scale.

0.5 pt 6. According to equation (3), what should be the hydraulic tortuosity in a straight pipe ?

0.5 pt 7. How are computed the error bars of figure 2 ?

0.5 pt 8. How is interpreted the regime change which appears in figure 2 ? At which Reynolds number does the transition occur ?

0.5 pt 9. What does figure 3 represent ?

0.5 pt 10. What is the paradox pointed out by the authors after figure 3 ?

0.5 pt 11. What is channeling effect ?

0.25 pt 12. What is the physical origin of disappearance of channeling effect at high Reynolds number ?

E Conclusions

1 pt 1. What is the main conclusion of the authors about the inertia effects onset ?

0.5 pt 2. What did the author observed in terms of flow structure when the Reynolds numbers is increased ?