DÉPARTEMENT MFEE
3ème année Parcours FEP/M2 FEIP
Toulouse INP – ENSEEIHT

TRANSFERS IN POROUS MEDIA
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Examination

Instructions and recommendations

1. **You may read these instructions, but do not turn the page or begin the work until instructed.**
2. Lecture notes provided at the end of the class are allowed during the examination. You can also have your personal notes. Any electronic device is forbidden, including calculator.
3. Communication between students is not allowed.
4. **Please**, add a 10-square margin at the top of your first page, and a 4-square margin at the left or right of each page.
5. Answers can be given in English, French or Spanish. Language level will not be a notation criterion.
6. Time limit : 120 minutes. Please stop working when asked.
7. Detail all work and assumptions in order to get the maximal score. A clear and detailed redaction is required.
8. This examination contains two sections. The first one is an exercise. The second one is a guided analysis of an article. Score is balanced between these two parts.
9. This exam is *probably* too long for two hours, so do not worry if you do not finish it.
10. Consequently, you should split equitably your time between both parts of the exam.
11. You have already a printed version of the article.
12. The present document contains 7 pages.

Exercise 1 : Bio-reactive nutrient transport in a yeast clog 20 points

We consider a microchannel of width W, height h, and length L. The microchannel has a constriction of width w at its end. Before starting the experiment, a suspension containing yeast (a micro-organism) is flowed through the device. The yeast cells are trapped by the constriction and form a *clog*. When the clog is large enough (typically $100\,\mu$m) the flow of suspension is stopped and replaced by a flow of culture medium with a fixed pressure drop ΔP.

This culture medium contains nutrients and feeds the yeast cells and allow them to grow and divide after consuming the nutrients. Consequently, the clog grows by yeast proliferation. Figure 1 shows a photograph (taken using a microscope) of the clog at three different times. Flow is leftwards. Under each photograph, the result of an image analysis shows a field of displacement inside the clog. The displacement is directly related to yeast proliferation. If there is no displacement, there is no yeast proliferation. Note that displacement in the clog is additive. That means that the displacement of the extremity of the clog is the sum of all the displacements between the constriction and the considered point.

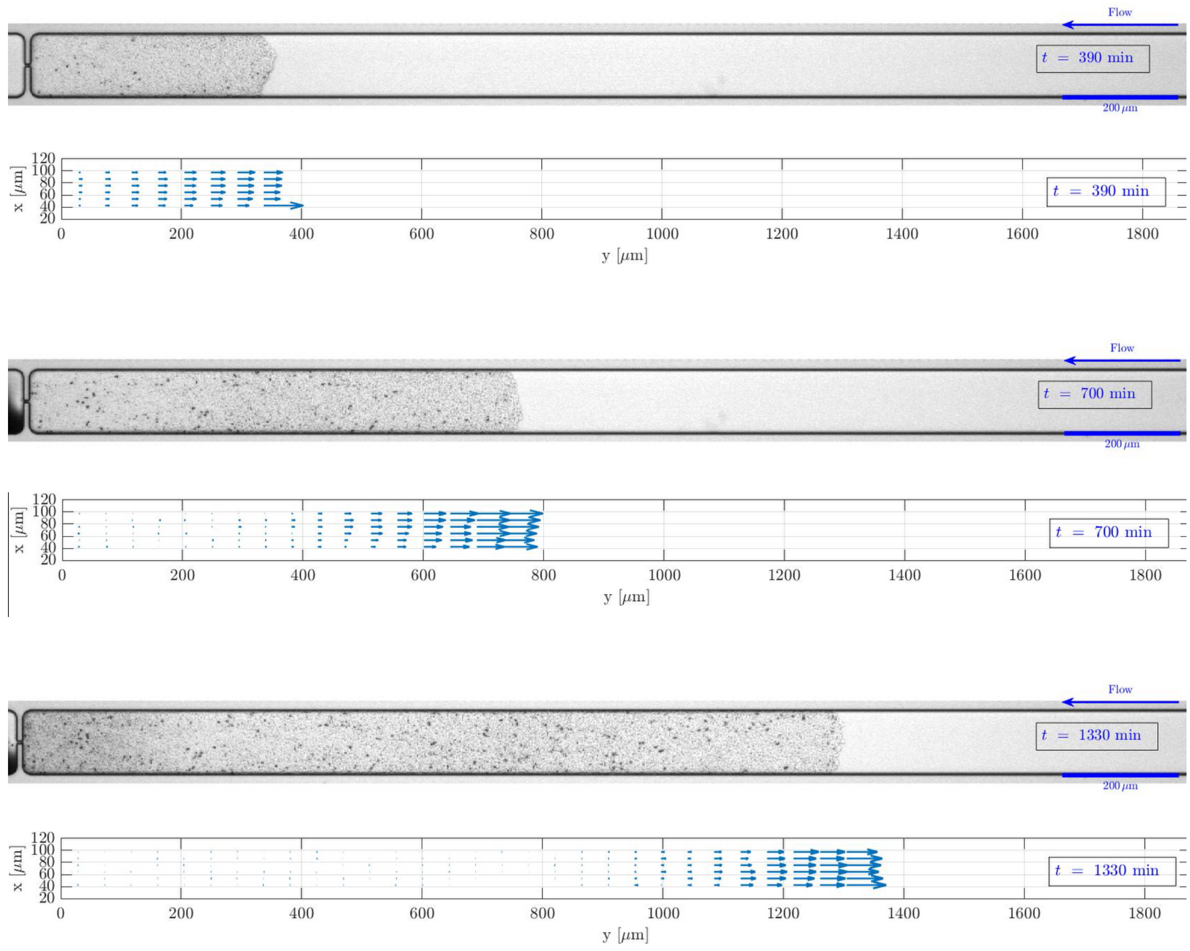

FIGURE 1 – Growth of a yeast clog in a microfluidic channel ending by a constriction. Flow is leftwards. For each pair of pictures, the top one is a photograph of the yeast clog, the bottom one is the displacement field in the clog at the same time. Three time points, indicated on each picture, are presented.

This problem is an example of bio-reactive feedback in a porous media. First, the clog grows. The hydraulic resistance of the clog increases as it grows. As the pressure drop is fixed, the flow rate decreases which decreases the nutrient availability. This finally has a negative impact on the ability of the clog to grow. Figure 2 shows a sketch of this feedback loop.

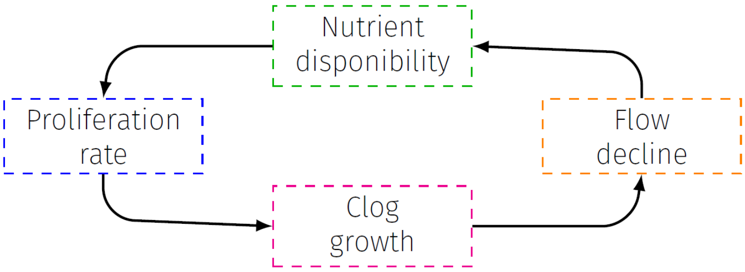

FIGURE 2 – Bio-reactive feedback loop in a growing yeast clog at fix pressure drop.

The aim of this problem is to model the clog growth with time. We consider a 1D model in the y direction. Figure 3 shows a sketch of the model. $Y(y,t)$ represents the yeast concentration, simply defined as a Heaviside function (in the clog the yeast concentration is fixed at Y_{max}, outside, it is fixed at 0). $L_c(t)$ is the clog's length. $V(t) = \langle v \rangle$ is the superficial average of the velocity, with v the local value of the velocity of the fluid (α-phase) in the clog. $C(y,t) = \langle C_{loc} \rangle^\alpha$ is the intrinsic average of the nutrient concentration in the α-phase, with C_{loc} the local value of the nutrient concentration.

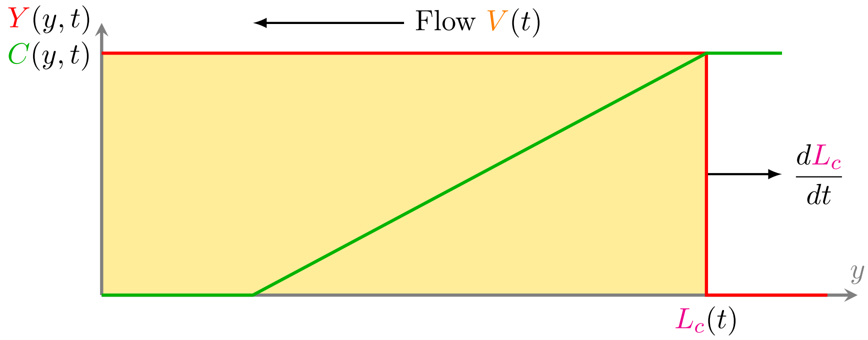

FIGURE 3 – Sketch of the problem used to define the physical quantities of the model. Flow is leftwards and the clog grows rightwards.

Darcy flow in the clog

0.5 pt 1. We assume that the flow through the clog is a Darcy flow. We note $R_h(t)$ the hydraulic resistance of the device {clog+ empty microchannel}. Recall the relation between $R_h(t)$, ΔP and the flow rate Q.

1 pt 2. We denote k the permeability of the clog, considered as an homogeneous porous media. The culture medium (which contains the nutrients necessary for clog proliferation) has a dynamic viscosity η. What is the relation between the clog's hydraulic resistance $R_c(t)$, k, L_c and other parameters of the problem ?

1.5 pt 3. The resistance of the empty device of length L is R_e. When the clog grows, the device fills up with yeast and a portion of the empty device is replaced by the porous medium formed by the clog. Show that we can write :

$$R_h(t) = \frac{\eta L_c(t)}{khW} + R_e \frac{L - L_c(t)}{L}. \tag{1}$$

1 pt 4. What is the condition on $L_c(t)/L$ for which the clog's hydraulic resistance dominates over the portion of the device remaining empty ? Numerically, this means that $L_c(t)/L \gg 0.009$. When you look at figure 1, do you think we can make this approximation ? Write the approximated expression of $R_h(t)$ that we will use for the rest of this problem.

Bio-reactive transport in the clog

0.5 pt 5. Recall the definition of $C(y,t) = \langle C_{loc} \rangle^\alpha$.

1 pt 6. We consider that nutrients are advected by the flow through the clog, and are dispersed in the porous media with a diffusivity tensor $\bar{\bar{D}}$. The nutrient consumption is modelled by a source term $\varepsilon Q_s(C)$. According to the lecture, write the general volume-averaged equation followed by $C(y,t)$.

1 pt 7. What are the mechanisms inducing dispersion in a porous media ? Detail each contribution.

2 pts 8. As the problem is 1D, the dispersion coefficient is a scalar D. Furthermore, the source term can be written $Q_s(C) = -\alpha \dfrac{dY}{dt}$ where α is a biological constant. We consider the Monod model for yeast proliferation such that

$$\frac{dY}{dt} = \mu_{max} \frac{C(y,t)}{C(y,t)+K} Y(y,t). \tag{2}$$

μ_{max} is the maximal proliferation rate of yeast cells and K the half-velocity constant. Using these indications, what is the specific form of the advection-reaction-dispersion equation you wrote in the previous question ? Please express the vector operators by their expressions in terms of partial derivatives. Be careful with the velocity orientation.

Asymptotic behaviours

In the following, we will assume that the clog's length variation is given by

$$\frac{dL_c}{dt} = \int_0^{L_c(t)} \mu_{max} \frac{C(y,t)}{C(y,t)+K} dy. \tag{3}$$

2 pts 9. We consider a short-time regime where all the yeast proliferate at their maximum rate, because the clog is short enough and all the clog is perfused by enough nutrients. This is equivalent to write, in the Monod Model, $C(y,t) \gg K\ \forall y$. We denote $L_{c0} = L_c(t=0)$ the initial clog length. Deduce the expression of $L_c(t)$ then $V(t)$ at the beginning of the experiment.

1.5 pt 10. After a long time, the velocity reaches a quasi-constant value V_{end}. This is due to the fact that $V(t)$ is hyperbolic in $L_c(t)$. Furthermore, the clog is separated in two zones : it is proliferating close to its extremity, and is not proliferating close to the constriction (see Figure 1). Under these assumptions, we can demonstrate (proof not required) that the length L_p of the proliferating zone is constant. To simplify we consider that in the proliferating zone, $C(y,t) \gg K$. Show that the asymptotic behaviours at high t of $L_c(t)$ and $V(t)$ are

$$L_c(t) = L_p \mu_{max} t + A_1 \quad \text{and} \quad V(t) =\propto 1 \mu_{max} L_p t + A_1, \tag{4}$$

where A_1 is a constant (not to be determined).

2 pts 11. Plot qualitatively $L_c(t)$ and $V(t)$ on the whole experiment time range. Comment.

Quasi-static resolution

The characteristic population doubling time for yeast cells is $\tau_{2\times} \approx 2.75\,\text{h}$. This is much higher than the typical advection time through the clog. Under this quasi-static assumption, we get the following advection-reaction-dispersion relation for a given time τ (the equation does not depend on t anymore) :

$$\frac{dC(y,\tau)}{dy} = \frac{\varepsilon \alpha \mu_{max} Y_{max}}{v(\tau)} \frac{C(y,\tau)}{C(y,\tau)+K}. \tag{5}$$

2 pts 12. By separating the variables, integrate this equation from y to L_c to get an implicit relation on the concentration $C(y,\tau)$. The concentration at $y = L_c(\tau)$ is C_{inj}.

1.5 pt 13. Using the normalizations $\tilde{C} = C/K$ and $\tilde{y} = y/L_c$, write the previous equation in dimensionless form, and show that a dimensionless number appears :

$$Da(\tau) = \left[\varepsilon \frac{\alpha \mu_{max} Y_{max} L_c(\tau)}{K V(\tau)}\right] \approx \varepsilon \frac{\alpha \mu_{max} Y_{max} \eta L_c^2(\tau)}{K \Delta P k}. \quad (6)$$

1 pt 14. This number is called the Damköhler number. Which physical mechanisms does it compare ?

1.5 pt 15. Figure 4 shows profiles of concentration at different Da. Comment the plot in light of the results and assumptions made previously in this problem.

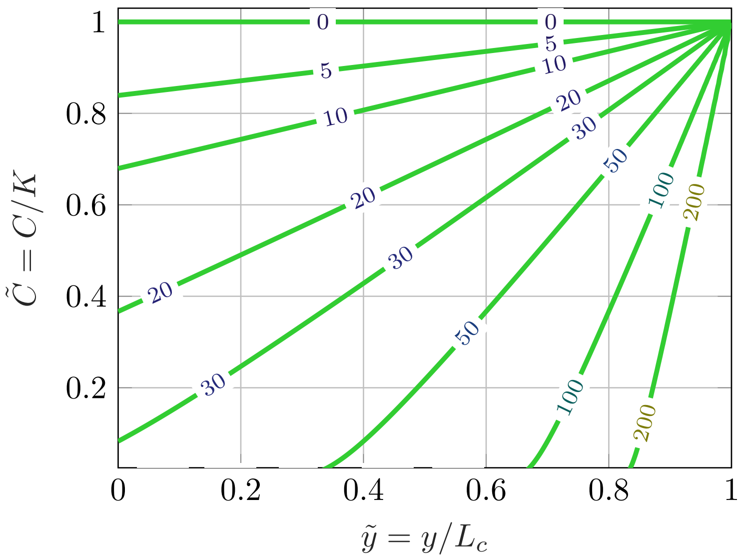

FIGURE 4 – Dimensionless concentration profiles in the clog for different Damköhler number indicated on each curve.

Exercise 2 : Article analysis 20 points

We propose to study an article entitled "Permeability of packs of polydisperse hard spheres" written by Vasseur *et al.* Before beginning the questions, have a quick overview of the article to review its structure and the main topic addressed. **You should have already read the article.** Your answers should be written in your own words ; do not paraphrase the article.

Questions

A Generalities about the article

0.25 pt 1. What is the year of publication of this article ?

0.25 pt 2. In which journal was it published ?

1.5 pt 3. In a few lines and without paraphrasing the text, summarize the objectives of the authors.

B Introduction and Background

0.5 pt 1. Give the name used in the lecture that corresponds to $\langle u \rangle$, and provide its mathematical definition.

0.75 pt 2. What other average definition did we use during the lecture ? What is its mathematical definition ? How does it relate to the previous average (question 1) ?

1.5 pt 3. In a few lines, without using mathematical formalism, explain how equation (1) is obtained.

0.25 pt 4. In the article, k is considered as a scalar. What does this imply for the porous media ?

0.25 pt 5. What is the name of the constant C in equation (2) of the article ?

0.5 pt 6. What are the two original articles which proposed the Kozeny-Carman relation ?

0.5 pt 7. According to the authors, what are the limits of the Kozeny-Carman relation ?

0.5 pt 8. To which type of packs of spheres does equation (3) refer ?

C Methods and sample geometry

C.1 Numerically generated samples

0.5 pt 1. Which approach was chosen by the authors to numerically generate polydisperse sphere packs ?

1 pt 2. Polydispersity is the fact of having spheres of different radii in the pack. Detail how the degree of polydispersity (polydispersivity) is computed in their work.

0.5 pt 3. What are the ranges of polydispersivity and porosity that the authors reach in their numerical porous media generation ?

1 pt 4. Propose an explanation for the impossibility of reaching very low porosity, and why the accessible porosity range depends on the polydispersivity.

0.75 pt 5. Demonstrate relation (6a).

C.2 Numerical simulations

0.25 pt 1. Which fluid do the authors simulate in the porous media ?

0.5 pt 2. Which phenomena could occur with such a fluid, and what dimensionless number is used to check it ?

1 pt 3. By observing the results of Figure 3 of the article, what first conclusion can you propose about the influence of polydispersivity on permeability ? Justify your answer.

0.25 pt 4. What is the range of Reynolds number used for the simulations ?

D Results and model analysis

1 pt 1. Describe the results presented in Figure 5.

1.5 pt 2. Propose a physical interpretation of these results.

1.75 pt 3. Establish relation (10) of the article.

1 pt 4. Why do the authors want to normalize their results of permeability using k_s ?

1.25 pt 5. Is the Kozeny-Carman relation a good model to take into account polydispersity in the permeability of sphere packs ? Justify your answer.

E Discussion and Conclusion

0.5 pt 1. In which porosity ranges are the Kozeny-Carman model and the cubic lattice packing model in good agreement with the data ?

0.5 pt 2. What do the authors conclude about the "percolation model" for polydisperse packed beds ?