
Exam correction

Exercise 1 : Bio-reactive transport in a yeast assembly

20 points

A Hydrodynamic resistance

- 1 pt 1. The flow rate in standard units is $Q = 5 \times 10^{-12} \text{ m}^3/\text{s}$. It corresponds to a mean velocity in the constriction of $U = 0.14 \text{ m/s}$. We then have :

$$Re = \frac{Uw}{\nu} = 0.4. \quad (1)$$

The flow is laminar.

- 1 pt 2. The Darcy's law for the laminar case is :

$$\langle \vec{v} \rangle = -\frac{\overline{\overline{K}}}{\mu} \text{grad} \langle P \rangle^\alpha. \quad (2)$$

$\overline{\overline{K}}$ is the permeability tensor and $\langle P \rangle^\alpha$ the pressure intrinsic average.

- 1 pt 3. Homogeneous porous medium. Negligible inertia. Local physics law valid at pore scale. Scale separation.
- 1.5 pt 4. The intrinsic average of a quantity G in the phase α is the average of this quantity on this phase only :

$$\langle G \rangle^\alpha = \frac{1}{V_\alpha} \int_{V_\alpha} G dV. \quad (3)$$

The superficial average is the average taken on the whole porous media :

$$\langle G \rangle = \frac{1}{V} \int_{V_\alpha} G dV. \quad (4)$$

So we have $\epsilon_\alpha \langle G \rangle^\alpha = \langle G \rangle$ where ϵ_α is the porosity (α -phase).

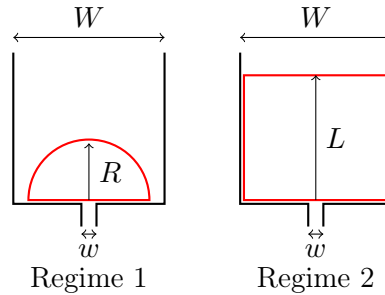
2 pts 5. We can base on the last slide of the Upscaling to porous media course. We have one microscopic, local relationship between a forcing term, a response term and the local media properties. We take volume average on the Representative Elementary Volume (REV), i.e. a volume whose porous structure and averaged physical quantity are uniform. With the appropriate gradient and divergence theorems, we obtain an average of the local equation, with interfacial integrals which needs to be estimated. To do that, we split the control and response variables as a sum of their mean on the REV and their fluctuation with respect to this mean. We can then obtain a second equation on the fluctuations. Hypotheses of homogeneity of the porous media and separation of the scales (pore, REV, macro) allow to simplify the equations. Finally, we use a closure relation linking linearly the fluctuations of a field at its mean in order to close the system and obtain an equation relating only the means of the forcing and response terms. We finally should obtain a relation between the averages of both control and response quantities, involving effective parameters.

1 pt 6. With inertia, we have the Darcy-Forchheimer's law :

$$\overrightarrow{\text{grad}}\langle P \rangle^\alpha = -\frac{\mu}{K}\langle \vec{v} \rangle - \rho\beta\langle \vec{v} \rangle^2, \tag{5}$$

where ρ is the density.

1 pt 7.



1.5 pt 8. We have the relation $Q = \|\langle \vec{v} \rangle\| \times (\pi r h)$ for any $w/2 < r < R$. Note that $\langle \vec{v} \rangle$ is along $-\vec{e}_r$. We can rewrite the Darcy's law projected along \vec{e}_r , with a scalar permeability :

$$-\frac{Q}{\pi r h} \vec{e}_r = -\frac{k}{\mu} \frac{dP}{dr} \vec{e}_r. \tag{6}$$

We separate the variables then integrate, considering that $\Delta P = P(R) - P(w/2)$:

$$\frac{Q}{\pi h} \int_{w/2}^R \frac{dr}{r} = \frac{k}{\mu} \int_{P(w/2)}^{P(R)} dP, \tag{7}$$

that leads to :

$$Q(R) = \frac{\pi k h}{\mu} \frac{\Delta P}{\ln(2R/w)}. \tag{8}$$

1 pt 9. We have the relation $Q = \|\langle \vec{v} \rangle\| \times (Wh)$. We can rewrite the Darcy's law projected along \vec{e}_z (vector which starts from the constriction). Note that $\langle \vec{v} \rangle$ is along $-\vec{e}_z$.

$$-\frac{Q}{Wh} \vec{e}_z = -\frac{k}{\mu} \frac{dP}{dz} \vec{e}_z. \tag{9}$$

We integrate :

$$\frac{Q}{Wh} \int_{W/2}^L dz = \frac{k}{\mu} \int_{P(W/2)}^{P(L)} dP. \tag{10}$$

Since $L \gg W$ we can consider that $P(L) - P(W/2) \approx \Delta P$:

$$Q(L) = \frac{Wkh}{\mu(L - W/2)} \Delta P. \quad (11)$$

1 pt 10. We could use the Kozeny-Carman relation :

$$k = \frac{\varepsilon^3 d_p^2}{180(1 - \varepsilon)^2}. \quad (12)$$

2.5 pts 11. Simulations show a dramatic increase of the hydraulic resistance for the very first particles arriving at the constriction. This is related to the sudden shrink of the possible flow paths at the constriction entrance. When the assembly grows, the hydraulic resistance carries on increasing, at a lower rate. This is related to the accumulation of new particles which increase the dimensions of the porous media.

The curve from Kozeny-Carman estimation shows a slightly increasing behaviour with respect to the number of particles in the assembly. Nevertheless, none sudden increase for $N = 3$ particles is observed, and the following increase rate is lower than the simulations' one.

Our system is quasi 2D for both experiments and simulations. One of the main assumption needed to derive the Darcy's law is scale separation. In our case, the microchannel depth is only $6 \mu\text{m}$ whereas the typical size of the particles is $4.5 \mu\text{m}$. Consequently, the typical pore dimension is about $1 \mu\text{m}$. Hence, the scale separation is not really checked in the depth direction. Furthermore, for the very first particles arriving at the constriction, this scale separation is not valid in all the spatial directions. To finish, Kozeny-Carman relation is computed for a monodisperse beads assembly, which is not the case here. Some corrections to Kozeny-Carman relation is necessary in this case.

B Nutrient transport and consumption

1 pt 1. With the standard notations we get :

$$\varepsilon \frac{\partial \langle C \rangle^\alpha}{\partial t} = \varepsilon \text{div}(\overline{\overline{D}} \cdot \overrightarrow{\text{grad}} \langle C \rangle^\alpha) - \text{div}(\langle \vec{v} \rangle \langle C \rangle^\alpha). \quad (13)$$

0.5 pt 2. As $\text{div}(\langle \vec{v} \rangle) = 0$ we get straightforwardly :

$$\varepsilon \frac{\partial \langle C \rangle^\alpha}{\partial t} = \varepsilon \text{div}(\overline{\overline{D}} \cdot \overrightarrow{\text{grad}} \langle C \rangle^\alpha) - \langle \vec{v} \rangle \cdot \overrightarrow{\text{grad}} \langle C \rangle^\alpha. \quad (14)$$

1.5 pt 3. We neglect the diffusion term as the Péclet number is high and there is not transverse dispersion as the problem is 1D. We neglect the temporal variation of the concentration (quasi-static). The nutrient consumption is a sink term. Moreover, it is a 1D problem. Finally we get :

$$V \frac{d \langle C \rangle^\alpha}{dz} = \alpha \left[\mu_{max} \frac{\langle C \rangle^\alpha}{\langle C \rangle^\alpha + K} \right] Y_m. \quad (15)$$

To be sure about the signs, remind that the concentration must decrease when you get closer and closer to the constriction. That means that the concentration gradient must be positive (it must increase when z increases).

1.5 pt 4. We set $C^* = \langle C \rangle^\alpha / C_{inj}$ and $z^* = z/L$. We have finally :

$$\frac{dC^*}{dz^*} = Da \frac{C^*}{C^* + K'}, \quad (16)$$

with $Da = \frac{\alpha L \mu_{max} Y_m}{V C_{inj}}$ and $K' = K/C_{inj}$.

- 1 pt 5. This dimensionless number compares advection to nutrient consumption. A high Da means a high consumption so the concentration profile will decrease rapidly from the assembly surface, whereas $Da \ll 1$ means that the advection is large enough to have an uniform nutrient concentration in the assembly.

Exercise 2 : Article analysis

20 points

Questions

A Generalities about the article

- 0.25 pt 1. 2017.
- 0.25 pt 2. Physical Review Fluids.
- 2 pts 3. The authors are interested in the mixing of a scalar in a 2D porous medium made of beads randomly stacked. They aim at understanding, experimentally, how a field of dye concentration disperses in this medium. To do this, an index matching is made between the porous medium and the liquid flowing through it, which makes it possible to monitor the concentration of dye during its advection and its dispersion in the medium.

B Introduction

- 1.5 pt 1. The first way is Brownian (or molecular) diffusion. There is also advection due to displacement fluid. There is also the Taylor-Aris dispersion due to shearing, and finally the mechanical dispersion due to the structure of the porous medium.
- 0.5 pt 2. It is the reference 17, published in Experiments in Fluids.
- 1 pt 3. The authors propose to focus more precisely on what happens at the pore scale, especially for the concentration field and the details of the mixing mechanisms. In effect, until now, studies were done on a macroscopic scale.

C Setup and Methods

- 1 pt 1. The problem of direct visualization in a porous medium is the presence of the solid structure. One way to illuminate and observe the entire porous medium is to match the optical index of the solid part and of the fluid. Beyond the transparency of the medium, this makes it possible to avoid shadow effects due to refraction, as shown in Figure 1.
- 0.5 pt 2. The water and the beads constituting the porous medium must have the same index, with a precision relative of the order of a 10^{-3} .
- 0.5 pt 3. Beads are hydrogels capable of absorbing water.
- 0.5 pt 4. Porosity is the ratio between the volume of the empty part of the porous medium and its volume : $\phi = V_\alpha/V$.
- 0.25 pt 5. Intrinsic velocity.
- 1 pt 6. The Péclet number represents the relationship between advection and diffusion phenomena. In the experiment presented, the Péclet number is of the order of 10^5 , therefore the molecular diffusion will be negligible.
- 0.25 pt 7. Two.

D Mixing Properties of the Flow

D.1 Plume dispersion

- 0.5 pt 1. Dispersion in a porous medium takes place in particular thanks to the combination of advection and mechanical dispersion due to the porous structure. Therefore, if the mean velocity field is anisotropic (here a preferred direction), the dispersion will be anisotropic, hence the need to define dispersion coefficients parallel to the flow, and perpendicular.
- 0.5 pt 2. We have $\delta^2(t) = s_0^2 + 4D_{\perp}x/v$. s_0 is the source width.

D.2 Growing sheets and necessary overlap

- 1 pt 1. It aims at fixing a length r , and at describing the contour of the sheet, at a distance x from the nozzle, using only segments of that length. The smaller r is, the more the description will be faithful.
- 1 pt 2. The proposed equation's solution is :

$$\left\langle \frac{\Delta r}{\Delta t} \right\rangle = A \exp(-r/d) + \gamma d. \quad (17)$$

If we place at $r \ll d$, the article tells that $\left\langle \frac{\Delta r}{\Delta t} \right\rangle \approx \gamma r$. Then we can develop the former expression and we get ($u = \gamma d$) :

$$A \left(1 + \frac{r}{d} \right) + \gamma d \approx \gamma r \Rightarrow A = -\gamma d, \quad (18)$$

and we find back the expression of the article.

- 1 pt 3. This expression represents the average of the speed increment between two material particles belonging to the contour of the plume. For low r , it is the shear that will create speed differences. Then it will tend exponentially towards a driven behaviour by the large-scale mean velocity.
- 3 pts 4. The length of the contour of the plume, and therefore its surface, grows exponentially. This leads to a thinning of the latter to an equilibrium thickness due to molecular diffusion. Although the volume of the plume increases, it remains restricted within a limited volume. There are therefore folding and superimposing of the plume on itself which greatly contributes to the dispersion.
- 0.5 pt 5. We can compare this type of dispersion with a turbulent flow, whereas here the flow has very little inertia.

D.3 Entangled fields

- 0.5 pt 1. Here the two nozzles inject two different dyes in order to understand how the two plumes mix.
- 1.5 pt 2. Figure 4 represents, for different depths in the medium, the joint probability of find a concentration C_1 and a concentration C_2 at the same point. Having a strong probability on the axes $C_1 = 0$ and $C_2 = 0$ shows the fact that there is no mixing yet, and therefore no correlation between the two concentrations. As you move through the porous media, the graphs tend towards a diagonal distribution of the joint probability which means a correlation between the two concentration fields.
- 1 pt 3. This shows that the two concentrations are not yet perfectly correlated.