Département MF2E
3ème année Parcours FEP/M2 FEIP
Toulouse INP – ENSEEIHT

Transfers in Porous Media
Olivier LIOT
2023-2024

Exam correction

Exercise 1 : Let's brew some coffee... 20 points

A Piston coffee maker

1 pt 1. In porous media physics, the saturation θ is defined as the ratio between the volume of water (or liquid) over the total volume of the porous medium $\theta = V_w/V$.

1 pt 2. $e = z\phi/(1-\varepsilon)$.

1 pt 3. V is the filtration velocity, which corresponds to the superficial average of the velocity in the porous media. $U = \langle v\rangle = \frac{1}{V}\int_{V_\alpha} v d\tau$. The intrinsic velocity writes $\langle v\rangle^\alpha = \frac{1}{V_\alpha}\int_{V_\alpha} v d\tau = \frac{\langle v\rangle}{\varepsilon}$.

1 pt 4. $U = \frac{dz}{dt}$ so $\frac{de}{dt} = U\phi/(1-\varepsilon) \Rightarrow e(t) = \frac{U\phi}{(1-\varepsilon)}t$ because $e(0) = 0$.

0.25 pt 5. $\varepsilon \approx 0.4$.

0.5 pt 6. $Re_p = \frac{\rho U d_p}{\varepsilon\eta} \approx 12.5$.

1.25 pt 7. The flow is driven by the Darcy-Forchheimer's law :

$$\overrightarrow{\text{grad}}\langle P\rangle^\alpha = -\eta\overline{\overline{K}}^{-1}\langle\overrightarrow{v}\rangle - \rho\beta\langle\overrightarrow{v}\rangle^2. \quad (1)$$

We can write it as :

$$\langle\overrightarrow{v}\rangle = -\frac{\overline{\overline{K}}_{app}}{\eta}\overrightarrow{\text{grad}}\langle P\rangle^\alpha \text{ with } K_{app} = \frac{\overline{\overline{K}}}{1+\epsilon\beta\sqrt{\overline{\overline{K}}}Re_p^*}, \quad (2)$$

where $Re_p^* = \frac{\rho\sqrt{\overline{\overline{K}}}||\langle\overrightarrow{v}\rangle||}{\eta\epsilon}$.

0.5 pt 8. We can use the Ergun's law as we consider the coffee grounds as a porous media made of monodisperse particles.

1.5 pt 9. We have the relation :

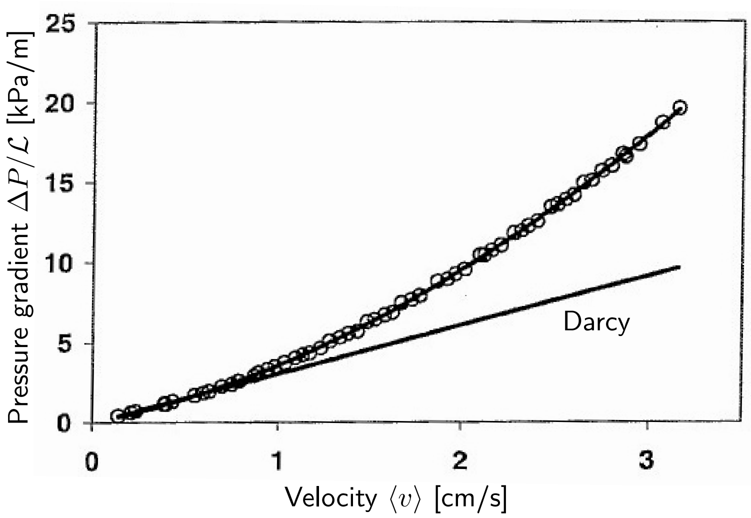
$$\frac{\Delta P}{e} = \frac{180(1-\epsilon)^2}{\epsilon^3}\frac{\eta}{D_p^2}U + \beta\frac{3\,(1-\epsilon)}{4\,\epsilon^3}\frac{\rho}{D_p}U^2, \tag{3}$$

where ΔP is the pressure drop across the porous media.

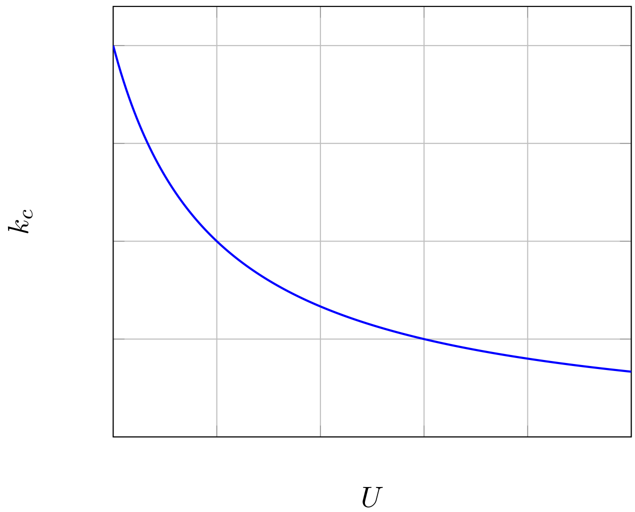
Given the Darcy-Forchheimer's law we have $k_c(U) = \dfrac{\eta U}{\Delta P}e$. With the Ergun's law we get :

$$k_c = \eta\left[\frac{180(1-\varepsilon)^2\eta}{\varepsilon^3 d_p^2} + \frac{3\beta(1-\varepsilon)\rho_w}{4\varepsilon^2 d_p}U\right]^{-1}. \tag{4}$$

1 pt 10. We have typically the following trend for $\Delta P/e = f(U)$.



The trend for $k_c(U)$ is :



This decrease is due to the appearance of recirculation cells in the porous media leading to a decrease of the effective section of the pores.

1 pt 11. According to the lecture (composite porous media), we can write :

$$k_{eff} = \frac{e(t)+L}{\frac{e(t)}{k_c(U)} + \frac{L}{k_p}}. \tag{5}$$

2 pts 12. The Darcy-Forchheimer's law can be written as :

$$\overrightarrow{\text{grad}}\langle P\rangle^{\alpha} = -\eta k_{eff}^{-1}\langle\overrightarrow{v}\rangle. \tag{6}$$

The definition of the hydraulic resistance is $R_h = \dfrac{\Delta P}{US}$. We consider a constant pressure gradient in the effective porous media. Thus we have $R_h = \dfrac{\eta}{k_{eff} S}(L+e)$. Finally, with the expression of $e(t)$, we obtain :

$$R_h(U,t) = \frac{\eta}{S}\left(\frac{L}{k_p} + \frac{\phi U}{\eta(1-\varepsilon)}\left[\frac{180(1-\varepsilon)^2\eta}{\varepsilon^3 d_p^2} + \frac{3\beta(1-\varepsilon)\rho_w}{4\varepsilon^2 d_p}U\right]t\right). \tag{7}$$

B An espresso ?

1 pt 1. The Darcy's law writes :

$$\langle \overrightarrow{v_\alpha} \rangle = -\frac{k_c}{\eta}\overrightarrow{\text{grad}}\langle P_\alpha \rangle^\alpha. \tag{8}$$

See the lecture for terms definition.

1 pt 2.

$$\epsilon \frac{\partial \langle C_\alpha \rangle^\alpha}{\partial t} = \epsilon \text{div}(\overline{\overline{D}} \cdot \overrightarrow{\text{grad}}\langle C_\alpha \rangle^\alpha) - \text{div}(\langle \overrightarrow{v} \rangle \langle C_\alpha \rangle^\alpha). \tag{9}$$

0.5 pt 3. Molecular diffusion, advection, Taylor-Aris dispersion and mechanical dispersion. See lectures.

1 pt 4. We consider only the z coordinate for vectors and variations. As the flow is incompressible, $\text{div}\langle \overrightarrow{v} \rangle = 0$. So we get :

$$\varepsilon \frac{\partial \langle C_\alpha \rangle^\alpha}{\partial t} = \frac{\partial}{\partial z}\left[\varepsilon D_{zz}\frac{\partial \langle C_\alpha \rangle^\alpha}{\partial z} - \langle v_\alpha \rangle_z \langle C_\alpha \rangle^\alpha\right]. \tag{10}$$

1.5 pt 5. We have $\langle C_\alpha \rangle^\alpha(z=0) = 0$ (no coffee in the water at the inlet). $\langle v_\alpha \rangle$ is uniform by flow rate conservation. We can integrate twice the equation 10 in steady-state. Using the boundary conditions, there is zero-concentration at the outlet (profile uniformly null along z). Source term is missing.

3 pts 6. A very basic model consists in considering that the coffee is dissolved in a homogeneous way on the coffee grounds with a source term σ whose dimension is concentration per time unit. We still consider steady-state and we assume that porosity (so permeability) is not affected by the dissolution. Then we get immediately :

$$0 = \frac{\partial}{\partial z}\left[\varepsilon D_{zz}\frac{\partial \langle C_\alpha \rangle^\alpha}{\partial z} - \langle v_\alpha \rangle_z \langle C_\alpha \rangle^\alpha\right] + \sigma. \tag{11}$$

As the permeability is uniform and using Darcy ($\langle v_\alpha \rangle_z = \dfrac{k_c}{\eta}\dfrac{\Delta P}{e}$), we get :

$$0 = \frac{\partial}{\partial z}\left[\varepsilon D_{zz}\frac{\partial \langle C_\alpha \rangle^\alpha}{\partial z} - \frac{k_c}{\eta}\frac{\Delta P}{e}\langle C_\alpha \rangle^\alpha\right] + \sigma. \tag{12}$$

We solve it by integrating twice and solving homogeneous and particular solutions for the second integration- (A and B are integrations constants) :

$$\langle C_\alpha \rangle^\alpha(z) = B\exp\left(\frac{k_c \Delta P}{\varepsilon \eta e D_{zz}}z\right) + \frac{\sigma \eta e}{k_c \Delta P}z + \frac{\varepsilon D_{zz}\sigma(\eta e)^2}{(k_c\Delta P)^2} - A\frac{\eta e}{k_c \Delta P}. \tag{13}$$

Using the boundary conditions which did not change, we get :

$$A = 0 \tag{14}$$

$$B = -\sigma D_{zz}\varepsilon\left(\frac{\eta e}{k_c \Delta P}\right)^2. \tag{15}$$

Finally,

$$\langle C_\alpha\rangle^\alpha(z) = \sigma\frac{\eta e}{k_c\Delta P}\left[-D_{zz}\varepsilon\left(\frac{\eta e}{k_c\Delta P}\right)\exp\left(\frac{k_c\Delta P}{\varepsilon\eta e D_{zz}}z\right) + z + D_{zz}\varepsilon\left(\frac{\eta e}{k_c\Delta P}\right)\right] \quad (16)$$

Exercise 2 : Article analysis 20 points

Questions

A Generalities about the article

0.25 pt 1. 2021.

0.25 pt 2. American Journal of Physics.

2 pts 3. The authors are interested in computing the force necessary to push the piston of a French plunger used to brew coffee. The use laboratory and "kitchen" experiments to determine this force and they relate it to the velocity of the piston. They also study the coffee grains structural properties. They finally compute the maximal force needed to operate in a given time. they also measure and estimate indirectly the coffee pack's porosity with a good accuracy.

B Introduction

1 pt 1. Fluid flow in a porous media, coffee dissolution and transport of a solute in a porous media (dispersion, diffusion).

0.5 pt 2. Mainly espressos coffee.

0.5 pt 3. "Environmental impacts of different methods of coffee preparation" (reference 8).

C Material characterization and methods

0.5 pt 1. $m = 54\,\mathrm{g}$.

1 pt 2. At-home experiments are performed at constant applied force with a mass disposed on the plunger. Then the displacement is measured over time. Laboratory experiments are made using an uniaxial press to measure the force for a fixed displacement velocity. In the laboratory experiments, the coffee pack thickness is measured over time during brewing process whereas in the at-home ones, it is measured only after the brewing time (5 min).

0.5 pt 3. The plunger velocity $\langle u\rangle$ is kept constant whereas the force F is measured.

1 pt 4. Coffee grains have a bimodal size distribution centred on $50\,\mu$m and $300\,\mu$m with an average of $103\,\mu$m. They are roughly spherical. There is an intra-grain porosity. The density of a coffee grain is about $\rho_c = 480\,\mathrm{kg/m^3}$.

0.25 pt 5. Gas (helium) pycnometry.

1 pt 6. Mercury porosimeter. As mercury has a large surface tension and is non-wetting on most of solids, there exists a high Laplace's pressure drop through a curved interface (meniscus). Consequently, a capillary pressure is needed for mercury to invade a pore of a given diameter. The smaller the pore size, the higher the capillary pressure. Mercury porosimeter push mercury in a chamber where the porous sample is placed. A ramp of pressure is applied. Mercury will invade the pores while pressure increases. The volume of pores filled for a given pressure finally gives the pore size distribution and the porosity.

D Results and Analysis

0.5 pt 1. 50 s.

1.25 pt 2. $\phi = 1 - \dfrac{\rho}{\rho_c}$ with article notations. $\rho = m/V = \dfrac{m}{\pi B^2 L}$. The measurements of ρ is extracted from a linear regression in figure 3 (b). Finally, $\phi = 1 - \dfrac{230}{480} \approx 0.53$.

1 pt 3. As all the coffee pack is already stacked under the plunger mesh, the hydraulic resistance of the coffee pack is about constant as the plunger is forcing down. Consequently, for a given applied force, the plunger velocity is constant (displacement lienar with time). Furthermore, as m increases, the coffee pack thickness increases and its hydraulic resistance too. This leads to a decrease on plunger velocity (for a given m_a) so on the $d(t)$ proportionality factor.

1 pt 4. It is probably due to rearrangements of the porous structure during the first centimetre of displacement : the grains can find a more compact spatial organization. This leads to less permeable porous media and so a higher operating force to move the plunger.

E Coffee percolation scaling

1 pt 1. Hydrodynamic resistance of the coffee pack, hydrodynamic resistance of the plunger's mesh, coffee grains buoyancy force, frictional force between the plunger and the jug, and plunger mass. They are all negligible excepted the hydrodynamic resistance due to the coffee pack.

1 pt 2. For one grain we have a buoyancy force $F_b = (\rho_f - \rho_c) g V_p$. Given the porosity definition, $\phi = V_{fluid}/V = 1 - nV_p/v$ where n is the number of grains. As the total buoyancy force is $F_B = nF_b$ we get equation 3 :

$$F_B = nF_b = (\rho_f - \rho_c) g V (1 - \phi). \tag{17}$$

2 pts 3. We can base on the slide 10 of the *Upscaling to porous media* course. We have one microscopic, local relationship between a forcing term, a response term and the local media properties. Here it is a Stokes equation coupled to incompressibility. We take volume average on the Representative Elementary Volume (REV), i.e. a volume whose porous structure and averaged physical quantity are uniform. With the appropriate gradient and divergence theorems, we obtain an average of the Stokes and continuity equations, with interfacial integrals which need to be estimated. To do that, we split the control and response variables as a sum of their average on the REV and their fluctuations with respect to this average. We can then obtain a second equation on the fluctuations. Hypotheses of homogeneity of the porous media and separation of the scales (pore, REV, macro) allow to simplify the equations. Then we use a closure relation linking linearly the fluctuations of a field to its average in order to close the system and obtain an equation relating only the means of the forcing and response terms. In the case of fluid flow, a Brinkman correction remains. Using the scale separation, we can neglect it and we get the Darcy's law.

0.5 pt 4. According to equation (6) we have a linear relation between F_h and the mass of coffee m.

F The permeability of packed coffee

1 pt 1. The velocity is extracted from the linear regression : $\langle u \rangle \approx 0.5\,\mathrm{mm/s}$. As $k = \langle u \rangle m \mu_f / (g \rho m_a)$, we compute $k \approx 2 \times 10^{-10}\,\mathrm{m}^2$.

1 pt 2. Flows in porous media lead to very inhomogeneous velocity fields. The complexity of the porous structure do not allow to compute analytically or numerically the full velocity field. That's why we need to use methods to upscale and homogenize the physical quantities, including velocity. The resultant averages (superficial and intrinsic) are the relevant quantities to work on the velocity in a porous media.

1 pt 3. Due to roughness of the coffee grains, their real radius are not the relevant as the specific surface will be higher in the case of a rough particle. To counter-balance this effect, the effective radius must be reduced in the K-C relation. Authors use the Sauter diameter.