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# Thermal transfer in porous media

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## Exercises

### 1 Derivation of heat equation

We consider a media at rest where the thermal transfer is made only by conduction. We remind the Fourier's law:

$$\vec{\phi} = -\bar{\lambda} \cdot \overrightarrow{\text{grad}}T, \quad (1)$$

with  $\vec{\phi}$  the heat flux density vector,  $\lambda$  the thermal conductivity tensor and  $T$  the temperature.

1. We consider that there is no mechanical work in the media. Write the first thermodynamic principle in the media (we denote as  $U$  the internal energy and  $Q$  the heat exchange during time  $\Delta t$ ).
2. We denote as  $m$  and  $c_v$  the total mass and specific heat capacity of the media at constant volume. Detail the expression of the internal energy variation  $\Delta U$  as a function of  $T$  and give the new expression of the first principle.
3. Write the internal energy balance (no source) in an elementary volume.
4. Using Fourier's law and the relation between  $U$  and  $T$ , give the heat equation.

### 2 Estimation of effective conductivity bounds

A porous media composed of two phases and crossed by a thermal flux can be simplified in two ways. First we consider that it is a succession of solid and liquid layers in the heat flux direction. Then the layers are parallel to the heat flux. These two models are a large simplification of a porous media and correspond to bounds for effective thermal conductivity estimation. We denote as  $\vec{\phi}$  the flux density vector crossing the porous media. The two layers are a fluid (conductivity  $\lambda_\alpha$ ) and a solid (conductivity  $\lambda_\sigma$ ).

1. Compute the effective conductivity in the first (series arrangement) case.

2. Compute the effective conductivity in the second (parallel arrangement) case.

### 3 Dimensionless numbers for convection in porous media

This exercise is part of adapted from an exercise of O. Thual. Let us consider a saturated porous media included between two horizontal plates in  $z = 0$  and  $z = d$  at fixed temperatures  $T_1$  and  $T_2$  respectively,  $T_1 > T_2$ . The velocity field is  $\vec{U} = \langle \vec{v} \rangle = (u, v, w)$ , pressure is denoted as  $P = \langle P_\alpha \rangle^\alpha$ , temperature as  $T = \langle T_\alpha \rangle^\alpha$ . These quantities are linked via the model equations:

$$\operatorname{div} \vec{U} = 0 \quad (2)$$

$$\rho = \rho(T_0)[1 - \beta(T - T_0)] \quad (3)$$

$$\frac{d\vec{U}}{dt} = -\frac{1}{\rho(T_0)} \overrightarrow{\operatorname{grad}} P - \frac{\rho}{\rho(T_0)} g \vec{e}_z - \frac{\nu}{k} \vec{U} \quad (4)$$

$$\frac{dT}{dt} = \kappa \Delta T. \quad (5)$$

$\rho$  is the density of the fluid phase,  $\beta$  the thermal expansivity coefficient,  $T_0$  a reference temperature,  $g$  the acceleration due to gravity,  $\vec{e}_z$  the unit vertical vector,  $k$  the porous media permeability,  $\nu$  the kinematic viscosity and  $D = \frac{\lambda}{(\rho c_v)^*}$  the equivalent thermal diffusivity of the porous media. The boundary conditions are  $T = T_1$  in  $z = 0$ ,  $T = T_2$  in  $z = d$  and  $w = 0$  in  $z = 0, d$ .

1. Determine the conductive temperature profile  $T_c(z)$  when the fluid is at rest. Deduce the pressure profile  $P_c(z)$  using the condition  $P_c(0) = P_0$ . We will note  $\Gamma = (T_1 - T_2)/d$ .

2. We note  $\theta = T - T_c$  and  $P = P_c + \rho(T_0)\Pi$ . All these quantities, except  $\rho(T_0)$ , depends on  $x, y, z, t$ . Write the model equations as a function of  $\vec{U}$  and these new fields.

3. We consider small fluctuations around the equilibrium state ( $\vec{U} = \vec{0}$ ). Show that the equations become:

$$\operatorname{div} \vec{U} = 0 \quad (6)$$

$$\frac{\partial \vec{U}}{\partial t} = -\overrightarrow{\operatorname{grad}} \Pi + \beta g \theta \vec{e}_z - \frac{\nu}{k} \vec{U} \quad (7)$$

$$\frac{\partial \theta}{\partial t} = \Gamma w + D \Delta \theta. \quad (8)$$

4. To have dimensionless equations we fix the typical scales:

$$[L] = d, [\tau] = d^2/D, [\Theta] = T_1 - T_2, [U] = [L]/[\tau] \text{ and } [\Pi] = [U^2]. \quad (9)$$

Write the dimensionless equations of the linearized model using these relations. Two dimensionless numbers should appear:

$$Pr = \frac{\nu d^2}{kD} \quad \text{and} \quad Ra = \frac{\beta g d k (T_1 - T_2)}{D \nu}. \quad (10)$$