Question 10

Mass conservation

porosity

$$\mathrm{div}(\rho\vec{v}) = -\phi\frac{\partial\rho}{\partial t}$$ $\qquad \langle\vec{v}\rangle^{*}\phi = \langle\vec{v}\rangle$

Darcy: $\vec{v} = -\frac{k_g}{\mu}\left(\overrightarrow{\mathrm{grad}}\,P - \cancel{\rho\vec{g}}\right)$ → neglect.

and ideal gas: $\rho = \frac{M_g P}{RT}$

$$\Rightarrow \mathrm{div}\left(\cancel{\frac{M_g}{RT}} P \times \left(-\frac{k_g}{\mu}\overrightarrow{\mathrm{grad}}\,P\right)\right) = -\phi\frac{\partial \cancel{\frac{M_g}{RT}} P}{\partial t}$$

and $k_g = k_\infty\left(1+\frac{b}{P}\right)$

$$\Rightarrow \mathrm{div}\left(P\left(1+\frac{b}{P}\right)\frac{k_\infty}{\mu}\overrightarrow{\mathrm{grad}}\,P\right) = \phi\frac{\partial P}{\partial t}.$$

$$\Rightarrow \boxed{\mathrm{div}\left(\frac{k_\infty\beta}{\mu}(P+b)\,\overrightarrow{\mathrm{grad}}\,P\right) = \phi\beta\frac{\partial P}{\partial t}}$$ $\qquad \beta = \frac{M_g}{RT}$

$P_k = P+b$

$$\Rightarrow \mathrm{div}\left(\frac{k_\infty\beta}{\mu}P_k\,\overrightarrow{\mathrm{grad}}\,P_k\right) = \phi\beta\frac{\partial P_k}{\partial t}$$

and $P_k\,\overrightarrow{\mathrm{grad}}\,P_k = \frac{1}{2}\overrightarrow{\mathrm{grad}}\,P_k^2$

and $P_k\frac{\partial P_k}{\partial t} = \frac{1}{2}\frac{\partial P_k^2}{\partial t}$

$$\Rightarrow \frac{1}{2}\mathrm{div}\left(\overrightarrow{\mathrm{grad}}\,P_k^2\right) = \frac{\mu}{k_\infty}\phi\frac{P_k}{P_k}\frac{\partial P_k}{\partial t}.$$

$$\Leftrightarrow \boxed{\mathrm{div}\left(\overrightarrow{\mathrm{grad}}\,P_k^2\right) = \frac{1}{\alpha}\frac{\partial P_k^2}{\partial t^2}} \quad \text{with } \alpha = \frac{k_\infty P_k}{\phi\mu} \qquad (2.1)$$

Question 1b

→ steady-state flow $\Rightarrow \frac{\partial}{\partial t} = 0$

$\Rightarrow \mathrm{div}\left(\overrightarrow{\mathrm{grad}}\,P_k^2\right) = 0 \Leftrightarrow \mathrm{div}\left(\frac{k_\infty\beta}{\mu}\overrightarrow{\mathrm{grad}}\,P_k^2\right) = 0$

1-dimensional $\Rightarrow \overrightarrow{\mathrm{grad}}\,P_k^2 = \begin{vmatrix}\frac{\partial P_k^2}{\partial x}\\ 0\\ 0\end{vmatrix} = \begin{vmatrix}2P_k\frac{\partial P_k}{\partial x}\\ 0\\ 0\end{vmatrix}$

$$\mathrm{div}\left(\frac{k_\infty\beta}{\mu}(P+b)\frac{\partial P}{\partial x}\vec{e}_x\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{k_\infty\beta}{\mu}(P+b)\frac{\partial P}{\partial x}\right) = 0 \qquad (3.1)$$

Resolution

$$\frac{k_\infty\beta}{\mu}(P+b)\frac{\partial P}{\partial x} = A$$

$$\Leftrightarrow (P+b)\frac{\partial(P+b)}{\partial x} = \frac{\mu A}{k_\infty\beta}$$

$$\frac{1}{2}\frac{\partial(P+b)^2}{\partial x} = \frac{\mu A}{k_\infty\beta} \Rightarrow (P+b)^2 = \frac{2\mu A}{k_\infty\beta}x + C$$

→ BC: $P(x=L) = P_L$

$\Rightarrow C = (P_L+b)^2 - \frac{2\mu A}{k_\infty\beta}L$

A? BC: at $x=0$ mass injection rate / unit of surface

$q_m = \frac{1}{S}\iint \rho(0)v(0)\,dS \qquad (\text{in } x=0)$

$q_m = \rho(0)v(0) = \beta P(0)\,v(0).$

$v(0)$?

$\vec{v} = -\frac{k_g}{\mu}\overrightarrow{\mathrm{grad}}\,P.$

and $P = -b + \sqrt{\frac{2\mu A}{k_\infty\beta}(x-L) + (P_L+b)^2}$

$$\overrightarrow{\mathrm{grad}}\,P\cdot\vec{e}_x = \frac{\partial P}{\partial x} = \frac{1}{2}\left(\frac{2\mu}{k_\infty\beta}A\right)\times\frac{1}{\sqrt{\frac{2\mu A}{k_\infty\beta}(x-L)+(P_L+b)^2}}$$

$$\Rightarrow v = -\frac{\cancel{k_\infty}}{\cancel{\mu}}\left(1+\frac{b}{P}\right)\frac{\cancel{\mu}}{\cancel{k_\infty}\beta}A\times\frac{1}{P+b}$$

$$\Rightarrow v = -A\frac{1+\frac{b}{P}}{P+b}\times\frac{1}{\beta}$$

$$\text{So } q_m = \beta P(0)v(0) = \cancel{\beta}\cancel{P(0)}\left(-\frac{A}{\cancel{\beta}}\frac{1+\frac{b}{P(0)}}{P(0)+b}\right)$$

$$\Rightarrow q_m = -A$$

$\Rightarrow A = -q_m$

$$\boxed{P(x) = -b + \sqrt{(P_L+b)^2 + \frac{2q_m\mu}{k_\infty\beta}(L-x)}} \qquad (3.2)$$

Question 1b

steady flow and mass conservation:

$\mathrm{div}(\rho\vec{v}) = 0$

$\Rightarrow \mathrm{div}\left(\beta P\left(-\frac{\bar{\bar{k}}_g}{\mu}\overrightarrow{\mathrm{grad}}(P)\right)\right) = 0$

and $\bar{\bar{k}}_g = \bar{\bar{k}}_\infty\left(1+\frac{b}{P}\right)$ $\quad \bar{\bar{k}}_\infty = \begin{pmatrix}k_{\infty,r} & 0\\ 0 & k_{\infty,z}\end{pmatrix}$ $\quad \bar{\bar{k}}_\infty = \begin{pmatrix}k_{\infty,r} & 0 & 0\\ 0 & k_{\infty,\theta} & 0\\ 0 & 0 & k_{\infty,z}\end{pmatrix}$

$$\Rightarrow \mathrm{div}\left(\frac{\beta P}{\mu}\underset{1+\frac{b}{P}}{\wedge}\begin{pmatrix}k_{\infty,r} & 0\\ 0 & k_{\infty,z}\end{pmatrix}\begin{vmatrix}\frac{\partial P}{\partial r}\\ \frac{\partial P}{\partial z}\end{vmatrix}\right) = 0$$

$$\Rightarrow \mathrm{div}\left(P_k\begin{vmatrix}k_{r,\infty}\frac{\partial P_k}{\partial r}\\ 0\\ k_{z,\infty}\frac{\partial P_k}{\partial z}\end{vmatrix}\right) = 0$$

$$\Leftrightarrow \mathrm{div}\left(\frac{1}{2}\begin{vmatrix}k_{r,\infty}\frac{\partial P_k^2}{\partial r}\\ 0\\ k_{z,\infty}\frac{\partial P_k^2}{\partial z}\end{vmatrix}\right) = 0$$

$\Phi = (P+b)^2 = P_k^2$

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\,k_{r,\infty}\frac{\partial\Phi}{\partial r}\right) + \frac{\partial}{\partial z}\left(k_{z,\infty}\frac{\partial\Phi}{\partial z}\right) = 0$$

$$\Leftrightarrow \boxed{k_{r,\infty}\left(\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r}\right) + k_{z,\infty}\frac{\partial^2\Phi}{\partial z^2} = 0} \qquad (3.3)$$