Darcy's law demonstration

1) Momentum equation: $\eta \Delta \vec{v} = \vec{\text{grad}}\, P$.
Continuity equation: $\text{div}\, \vec{v} = 0$.

1) BC: $\vec{v} = \vec{0}$ at $S_{s\sigma}$
$\vec{v} = \vec{v}_i(\vec{x}, t)$ at $S_{s\ell}$

3) $\langle \text{div}\, \vec{v} \rangle = 0$,
$0 = \text{div} \langle \vec{v} \rangle + \frac{1}{V} \int_{S_{s\sigma}} \vec{v} \cdot \vec{n}_{s\sigma}\, dS$. ($\vec{v} \to \vec{0}$)

$\Rightarrow \boxed{\text{div} \langle \vec{v} \rangle = 0}$

But $\langle \vec{v} \rangle = \varepsilon_s \langle \vec{v} \rangle^s$
$\Rightarrow \text{div}\left[\varepsilon_s \langle \vec{v} \rangle^s\right] = 0$
and $\text{div}(f \vec{A}) = f\, \text{div}\, \vec{A} + \vec{\text{grad}}\, f \cdot \vec{A}$.
$\Rightarrow \varepsilon_s\, \text{div} \langle \vec{v} \rangle^s + \vec{\text{grad}}\, \varepsilon_s \cdot \langle \vec{v} \rangle^s = 0$
$\Rightarrow \boxed{\text{div} \langle \vec{v} \rangle^s = -\varepsilon_s^{-1}\, \vec{\text{grad}}\, \varepsilon_s \cdot \langle \vec{v} \rangle^s}$

4) $\langle \vec{\text{grad}}\, P \rangle = \vec{\text{grad}} \langle P \rangle + \frac{1}{V} \int_{S_{s\sigma}} P \vec{n}_{s\sigma}\, dS$.
and $\langle P \rangle = \varepsilon_s \langle P \rangle^s$
$\Rightarrow \langle \vec{\text{grad}}\, P \rangle = \varepsilon_s\, \vec{\text{grad}} \langle P \rangle^s + \langle P \rangle^s\, \vec{\text{grad}}\, \varepsilon_s + \frac{1}{V} \int_{S_{s\sigma}} P \vec{n}_{s\sigma}\, dS$.

5) $P = \langle P \rangle^s + \tilde{P}$
and $\frac{1}{V} \int_{S_{s\sigma}} P \vec{n}_{s\sigma}\, dS = -\langle P \rangle^s\, \vec{\text{grad}}\, \varepsilon_s + \frac{1}{V} \int_{S_{s\sigma}} \tilde{P} \vec{n}_{s\sigma}\, dS$ ($\to 0$)
$\Rightarrow \boxed{\langle \vec{\text{grad}}\, P \rangle = \varepsilon_s\, \vec{\text{grad}} \langle P \rangle^s + \frac{1}{V} \int_{S_{s\sigma}} \tilde{P} \vec{n}_{s\sigma}\, dS}$.

Flow rate through a assembly of cylinders

1) One cylinder $\langle v \rangle = \frac{D^2\, \Delta P}{32 \eta L}$
2) $q = S \times \langle v \rangle = \pi \frac{D^2}{4} \times \langle v \rangle = \frac{\pi D^4\, \Delta P}{128 \eta L}$
3) We know $V \to$ total porous medium volume.
$U = \langle v \rangle \to$ superficial velocity
$Q = US = vq$
$U = \frac{vq}{S} = \frac{q}{S} \frac{\pi D^4\, \Delta P}{128 \eta}$ ($S = V$)

4) one cylinder: volume: $L \pi \frac{D^2}{4}$
$\Rightarrow \varepsilon = \frac{\text{void volume}}{V} = \frac{n}{V} \pi L \frac{D^2}{4}$

5) Specific surface: $a = \frac{S_{s\sigma}}{V} = \frac{n \pi D L}{V}$

6) $U = \frac{n}{S} \frac{\pi D^4 \Delta P}{128 \eta}$
$U = \frac{n}{V} \frac{\pi D^4 \Delta P}{128 \eta} = \frac{\varepsilon D^2}{32 L \eta} \Delta P$
$U = \frac{\varepsilon^3}{2 a^2} \frac{\Delta P}{\eta L}$

U: filtration velocity
$\Rightarrow$ superficial average of the velocity.
$\left(\langle v \rangle = \frac{1}{V} \int_{V_v} v\, dV \right)$

7) The length of the cylinders $L' \simeq L$
$\Rightarrow U' = \frac{U}{\varepsilon^2}$

$\langle \Delta \vec{v} \rangle = \langle \text{div}(\vec{\text{grad}}\, \vec{v}) \rangle$
↳ gradient of a vector
$\langle \Delta \vec{v} \rangle = \text{div} \langle \vec{\text{grad}}\, \vec{v} \rangle + \frac{1}{V} \int_{S_{s\sigma}} \vec{\text{grad}}\, \vec{v} \cdot \vec{n}_{s\sigma}\, dS$
$\Rightarrow \langle \vec{v} \rangle = \langle \vec{v} \rangle^s + \tilde{\vec{v}}$
$\Rightarrow \langle \Delta \vec{v} \rangle = \Delta \langle \vec{v} \rangle + \text{div}\left(\frac{1}{V} \int_{S_{s\sigma}} \vec{\text{grad}}\, \tilde{\vec{v}} \cdot \vec{n}_{s\sigma}\, dS \right)$
$- \vec{\text{grad}}\, \varepsilon_s \cdot \vec{\text{grad}} \langle \vec{v} \rangle^s + \frac{1}{V} \int_{S_{s\sigma}} \vec{\text{grad}}\, \tilde{\vec{v}} \cdot \vec{n}_{s\sigma}\, dS$

Averaged momentum equation:
$\varepsilon_s\, \vec{\text{grad}} \langle P \rangle^s + \frac{1}{V} \int_{S_{s\sigma}} \tilde{P} \vec{n}_{s\sigma}\, dS = \eta \left[\Delta \langle \vec{v} \rangle - \vec{\text{grad}}\, \varepsilon_s \cdot \vec{\text{grad}} \langle \vec{v} \rangle^s + \frac{1}{V} \int_{S_{s\sigma}} \vec{\text{grad}}\, \tilde{\vec{v}} \cdot \vec{n}_{s\sigma}\, dS \right]$

HYPOTHESIS
uniform porosity! $\Rightarrow \vec{\text{grad}}\, \varepsilon_s = \vec{0}$
and (small) $\langle \vec{v} \rangle = \varepsilon_s \langle \vec{v} \rangle^s$
$\Rightarrow \varepsilon_s\, \vec{\text{grad}} \langle P \rangle^s + \frac{1}{V} \int_{S_{s\sigma}} \tilde{P} \vec{n}_{s\sigma}\, dS = \eta \left[\varepsilon_s \Delta \langle \vec{v} \rangle + \frac{1}{V} \int_{S_{s\sigma}} \vec{\text{grad}}\, \tilde{\vec{v}} \cdot \vec{n}_{s\sigma}\, dS \right]$

intrinsic terms on fluctuations.

Equation on fluctuations.

at $S_{s\sigma} \to \tilde{\vec{v}} = -\langle \vec{v} \rangle^s$
because $\vec{v} = \langle \vec{v} \rangle^s + \tilde{\vec{v}} = \vec{0}$ at $S_{s\sigma}$

$\text{div} \langle \vec{v} \rangle^s = -\varepsilon_s^{-1}\, \vec{\text{grad}}\, \varepsilon_s \cdot \langle \vec{v} \rangle^s$
$\Rightarrow \text{div}(\vec{v} - \tilde{\vec{v}}) = -\varepsilon_s^{-1}\, \vec{\text{grad}}\, \varepsilon_s \cdot \langle \vec{v} \rangle^s$
$\boxed{\text{div}(\tilde{\vec{v}}) = \varepsilon_s^{-1}\, \vec{\text{grad}}\, \varepsilon_s \cdot \langle \vec{v} \rangle^s = 0}$ (continuity eq°) (uniform porosity)

Momentum: $\eta \Delta \vec{v} = \vec{\text{grad}}\, P$ (A) ($S + L_n$)
and: $\varepsilon_s\, \vec{\text{grad}} \langle P \rangle^s + \frac{1}{V} \int_{S_{s\sigma}} \tilde{P} \vec{n}_{s\sigma}\, dS = \eta \left[\varepsilon_s \Delta \langle \vec{v} \rangle^s + \frac{1}{V} \int_{S_{s\sigma}} \vec{\text{grad}}\, \tilde{\vec{v}} \cdot \vec{n}_{s\sigma}\, dS \right]$ (B)

(B) − ε_s(A) $\Rightarrow$
$\vec{\text{grad}}\, P - \varepsilon_s\, \vec{\text{grad}} \langle P \rangle^s - \frac{1}{V} \int_{S_{s\sigma}} \tilde{P} \vec{n}_{s\sigma}\, dS = \eta \left[\Delta \vec{v} - \varepsilon_s \Delta \langle \vec{v} \rangle^s - \frac{1}{V} \int_{S_{s\sigma}} \vec{\text{grad}}\, \tilde{\vec{v}} \cdot \vec{n}_{s\sigma}\, dS \right]$
$\vec{\text{grad}}\, (P - \varepsilon_s \langle P \rangle^s) - \frac{1}{V} \int_{S_{s\sigma}} \tilde{P} \vec{n}_{s\sigma}\, dS = \eta \left[\Delta (\vec{v} - \varepsilon_s \langle \vec{v} \rangle^s) - \frac{1}{V} \int_{S_{s\sigma}} \vec{\text{grad}}\, \tilde{\vec{v}} \cdot \vec{n}_{s\sigma}\, dS \right]$
$/\varepsilon_s$ and $V \varepsilon_s = V_s$
$\vec{\text{grad}}\, \tilde{P} - \frac{1}{V_s} \int_{S_{s\sigma}} \tilde{P} \vec{n}_{s\sigma}\, dS = \eta \left[\Delta \tilde{\vec{v}} - \frac{1}{V_s} \int_{S_{s\sigma}} \vec{\text{grad}}\, \tilde{\vec{v}} \cdot \vec{n}_{s\sigma}\, dS \right]$

CLOSURE

→ desire of a reference field (P or $\vec{v}$)
$\tilde{\vec{v}} = \bar{\bar{B}} \cdot \langle \vec{v} \rangle^s$ $\quad \bar{\bar{B}}$ is a tensor
$\tilde{P} = \eta\, \vec{b} \cdot \langle \vec{v} \rangle^s$ $\quad \vec{b}$ is a vector.
$\bar{\bar{B}}$ and $\vec{b}$ → depend only of porous medium geometry.

$\Rightarrow \vec{\text{grad}} \langle P \rangle^s = \eta\, \Delta \langle \vec{v} \rangle^s - \eta\, \varepsilon_s \bar{\bar{K}}^{-1} \langle \vec{v} \rangle^s$
with $-\varepsilon_s \bar{\bar{K}}^{-1} = \frac{1}{V_s} \int_{S_{s\sigma}} \vec{\text{grad}}\, \bar{\bar{B}} \cdot \vec{n}_{s\sigma}\, dS - \frac{1}{V_s} \int_{S_{s\sigma}} \vec{b} \cdot \vec{n}_{s\sigma}\, dS$.
(because on the integration domain $S_{s\sigma}$, $\langle \vec{v} \rangle^s$ is constant.
(hypothesis of scale separation).

$\Rightarrow \langle \vec{v} \rangle^s = -\frac{\bar{\bar{K}}}{\eta} \vec{\text{grad}} \langle P \rangle^s + \bar{\bar{K}}\, \Delta \langle \vec{v} \rangle^s$ Brinkman

$\bar{\bar{K}}$: permeability tensor.

Darcy's law with Brinkman correction.

orders of magnitude:
$\bar{\bar{K}} \sim l_c^2 \Rightarrow \bar{\bar{K}}\, \Delta \langle \vec{v} \rangle^s \sim \frac{l_c^2}{L_c^2} \frac{\langle \vec{v} \rangle}{\varepsilon_s}$
↳ spatial variation of the average velocity.

Scale separation $\Rightarrow l_c^2 \ll L_v^2$
$\to \varepsilon_s \sim 1$
$\Rightarrow \bar{\bar{K}}\, \Delta \langle \vec{v} \rangle^s \ll \langle \vec{v} \rangle$

Finally $\langle \vec{v} \rangle = -\frac{\bar{\bar{K}}}{\eta} \vec{\text{grad}} \langle P \rangle^s$ (Darcy)
↳ Darcy velocity or filtration velocity.

If we consider isotropic and uniform porous medium,
$\bar{\bar{K}} = K_{\text{scalar}}$
and flowrate: $Q = K A \frac{\Delta P}{L \eta}$
↳ how much?

Packed beds permeability.

1) Configuration (a)
each layer: q_i $\quad i = 1, 2$
$Q = \frac{N}{2} q_1 + \frac{N}{2} q_2$ and $q_i = k_i L h_i \frac{\Delta P}{\eta L}$
$Q = \frac{N}{2} \left(k_1 h_1 L \frac{\Delta P}{\eta L} + k_2 h_2 L \frac{\Delta P}{\eta L} \right)$
Moreover $A = L^2 = \frac{N}{2} L (h_1 + h_2) \Rightarrow \frac{N}{2} = \frac{L}{h_1 + h_2}$
$\Rightarrow Q = \underbrace{\frac{L}{h_1 + h_2} L}_{A} \underbrace{(k_1 h_1 + k_2 h_2)}_{k_\parallel} \frac{\Delta P}{\eta L}$

$\boxed{k_\parallel = \frac{k_1 h_1 + k_2 h_2}{h_1 + h_2}}$

Perpendicular configuration. (b)
$Q = q_1 = q_2$
$Q = k_i L^2 \frac{\delta P_i}{\eta h_i}$ and $\Delta P = \frac{N}{2} (\delta P_1 + \delta P_2)$
$q_1 + \frac{k_2 h_2}{k_1 h_1} q_2 = \frac{k_1 L^2}{\eta h_1} (\delta P_1 + \delta P_2)$
$\Rightarrow Q \left(1 + \frac{k_2 h_2}{k_1 h_1} \right) = \frac{k_1 L^2}{\eta h_1} \Delta P \frac{2}{N}$
and $\frac{N}{2} = \frac{L}{h_1 + h_2}$
$\Rightarrow Q = \frac{k_1 L^2}{\eta h_1} \Delta P \frac{(h_1 + h_2)}{L} \times \frac{1}{1 + \frac{k_2 h_2}{k_1 h_1}}$

$Q = \frac{k_1 h_1 (h_1 + h_2)}{k_1 h_1 + k_2 h_2} L^2 \frac{\Delta P}{\eta L}$
$k_\perp = \frac{h_1 + h_2}{h_1 / k_1 + h_2 / k_2}$