Darcy's law demonstration

1) Momentum equation : $\eta \Delta \vec{v} = \overrightarrow{\text{grad}}\, P.$

Continuity equation : $\text{div}\, \vec{v} = 0.$

2) BC : $\vec{v} = \vec{0}$ at $S_{\alpha\sigma}$

$\vec{v} = \vec{f}(\vec{x}, t)$ at $S_{\alpha e}$

3) $\langle \text{div}\, \vec{v} \rangle = 0$

$0 = \text{div} \langle \vec{v} \rangle + \frac{1}{V} \int_{S_{\alpha\sigma}} \vec{v} \cdot \vec{n}_{\alpha\sigma}\, dS. \rightarrow \vec{0}$

$\Rightarrow \boxed{\text{div} \langle \vec{v} \rangle = 0}$

But $\langle \vec{v} \rangle = \varepsilon_\alpha \langle \vec{v} \rangle^\alpha$

$\Rightarrow \text{div} \left[\varepsilon_\alpha \langle \vec{v} \rangle^\alpha \right] = 0$

and $\text{div}(f \vec{A}) = f\, \text{div}\, \vec{A} + \overrightarrow{\text{grad}}\, f \cdot \vec{A}.$

$\Rightarrow \varepsilon_\alpha\, \text{div} \langle \vec{v} \rangle^\alpha + \overrightarrow{\text{grad}}\, \varepsilon_\alpha \cdot \langle \vec{v} \rangle^\alpha = 0$

$\Rightarrow \boxed{\text{div} \langle \vec{v} \rangle^\alpha = - \varepsilon_\alpha^{-1}\, \overrightarrow{\text{grad}}\, \varepsilon_\alpha \cdot \langle \vec{v} \rangle^\alpha}$

4) $\langle \overrightarrow{\text{grad}}\, P \rangle = \overrightarrow{\text{grad}} \langle P \rangle + \frac{1}{V} \int_{S_{\alpha\sigma}} P \vec{n}_{\alpha\sigma}\, dS.$

and $\langle P \rangle = \varepsilon_\alpha \langle P \rangle^\alpha$

$\Rightarrow \langle \overrightarrow{\text{grad}}\, P \rangle = \varepsilon_\alpha \overrightarrow{\text{grad}} \langle P \rangle^\alpha + \langle P \rangle^\alpha \overrightarrow{\text{grad}}\, \varepsilon_\alpha + \frac{1}{V} \int_{S_{\alpha\sigma}} P \vec{n}_{\alpha\sigma}\, dS.$

$\rightarrow 0$

5) $P = \langle P \rangle^\alpha + \tilde{P}$

and $\frac{1}{V} \int_{S_{\alpha\sigma}} P \vec{n}_{\alpha\sigma}\, dS = - \langle P \rangle^\alpha\, \overrightarrow{\text{grad}}\, \varepsilon_\alpha + \frac{1}{V} \int_{S_{\alpha\sigma}} \tilde{P} \vec{n}_{\alpha\sigma}\, dS$

$\Rightarrow \boxed{\langle \overrightarrow{\text{grad}}\, P \rangle = \varepsilon_\alpha \overrightarrow{\text{grad}} \langle P \rangle^\alpha + \frac{1}{V} \int_{S_{\alpha\sigma}} \tilde{P} \vec{n}_{\alpha\sigma}\, dS.}$

$\rightarrow$ Viscous term : $\Delta \vec{v}.$

$\langle \Delta \vec{v} \rangle = \langle \text{div}(\underbrace{\overrightarrow{\text{grad}}\, \vec{v}}) \rangle$

$\hookrightarrow$ gradient of a vector $\Rightarrow$ tensor

$\langle \Delta \vec{v} \rangle = \underbrace{\text{div} \langle \overrightarrow{\text{grad}}\, \vec{v} \rangle}_{J} + \underbrace{\frac{1}{V} \int_{S_{\alpha\sigma}} \overrightarrow{\text{grad}}\, \vec{v} \cdot \vec{n}_{\alpha\sigma}\, dS.}_{I}$

$\rightarrow \vec{v} = \langle \vec{v} \rangle^\alpha + \tilde{\vec{v}}$

$I = - \overrightarrow{\text{grad}} \langle \vec{v} \rangle^\alpha \cdot \overrightarrow{\text{grad}}\, \varepsilon_\alpha + \frac{1}{V} \int_{S_{\alpha\sigma}} \overrightarrow{\text{grad}}\, \tilde{\vec{v}} \cdot \vec{n}_{\alpha\sigma}\, dS$

$J = \text{div}(\overrightarrow{\text{grad}} \langle \vec{v} \rangle) + \text{div}\left(\frac{1}{V} \int_{S_{\alpha\sigma}} \vec{v} \cdot \vec{n}_{\alpha\sigma}\, dS \right)$

$= \Delta \langle \vec{v} \rangle + \text{div}\left(\frac{1}{V} \int_{S_{\alpha\sigma}} \vec{v} \cdot \vec{n}_{\alpha\sigma}\, dS \right)$

$\Rightarrow \langle \Delta \vec{v} \rangle = \Delta \langle \vec{v} \rangle + \text{div}\left(\frac{1}{V} \int_{S_{\alpha\sigma}} \vec{v} \cdot \vec{n}_{\alpha\sigma}\, dS \right)$

$- \overrightarrow{\text{grad}}\, \varepsilon_\alpha \cdot \overrightarrow{\text{grad}} \langle \vec{v} \rangle^\alpha + \frac{1}{V} \int_{S_{\alpha\sigma}} \overrightarrow{\text{grad}}\, \tilde{\vec{v}} \cdot \vec{n}_{\alpha\sigma}\, dS$

Averaged momentum equation :

$\varepsilon_\alpha \overrightarrow{\text{grad}} \langle P \rangle^\alpha + \frac{1}{V} \int_{S_{\alpha\sigma}} \tilde{P} \vec{n}_{\alpha\sigma}\, dS = \eta \Big[\Delta \langle \vec{v} \rangle - \cancel{\overrightarrow{\text{grad}}\, \varepsilon_\alpha \cdot \overrightarrow{\text{grad}} \langle \vec{v} \rangle^\alpha}$

$+ \frac{1}{V} \int_{S_{\alpha\sigma}} \overrightarrow{\text{grad}}\, \tilde{\vec{v}} \cdot \vec{n}_{\alpha\sigma}\, dS \Big]$

HYPOTHESIS

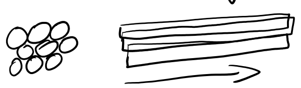
uniform porosity ! $\Rightarrow \overrightarrow{\text{grad}}\, \varepsilon_\alpha = \vec{0}$

and (remind) $\langle \vec{v} \rangle = \varepsilon_\alpha \langle \vec{v} \rangle_\alpha$

$\Rightarrow \varepsilon_\alpha \overrightarrow{\text{grad}} \langle P \rangle^\alpha + \frac{1}{V} \int_{S_{\alpha\sigma}} \tilde{P} \vec{n}_{\alpha\sigma}\, dS = \eta \left[\varepsilon_\alpha \Delta \langle v \rangle^\alpha + \frac{1}{V} \int_{S_{\alpha\sigma}} \overrightarrow{\text{grad}}\, \tilde{\vec{v}} \cdot \vec{n}_{\alpha\sigma}\, dS \right]$

interfacial terms on fluctuations.

Flow rate through an assembly of cylinders



Q

1) One cylinder $\langle v \rangle = \frac{D^2\, \Delta P}{32\, \eta L}$

2) $q = S \times \langle v \rangle = \pi \frac{D^2}{4} \times \langle v \rangle = \frac{\pi D^4\, \Delta P}{128\, \eta L}$

3) We know $V \rightarrow$ total porous medium volume.

$U = \langle v \rangle \rightarrow$ superficial velocity

$Q = US = nq$

$U = \frac{nq}{S} = \frac{n}{\underset{=V}{SL}} \frac{\pi D^4\, \Delta P}{128\, \eta}$

4) One cylinder : volume : $L \pi \frac{D^2}{4}$

$\Rightarrow \varepsilon = \frac{n \times \text{volume}}{V} = \frac{n}{V} \pi L \frac{D^2}{4}$

5) Specific surface : $a = \frac{S_{\alpha\sigma}}{V} = \frac{n \pi D L}{V}$

6) $U = \frac{n}{\underset{V}{SL}} \frac{\pi D^4 \Delta P}{128\, \eta}$

$U = \frac{n}{V} \frac{\pi D^4 \Delta P}{128\, \eta} = \frac{\varepsilon\, D^2\, \Delta P}{32\, L\, \eta}$

$U = \frac{\varepsilon^3}{2 a^2} \frac{\Delta P}{\eta L}$

U : filtration velocity

$\Leftrightarrow$ superficial average of the velocity.

$\left(\langle v \rangle = \frac{1}{V} \int_{V_\alpha} v\, dV \right)$

7) The length of the cylinders $L' = \tau L$

$\Rightarrow U' = \frac{U}{\tau^2}$