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HETEROGENEOUS MEDIUM. IS AN EQUIVALENT MACROSCOPIC DESCRIPTION POSSIBLE?

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Abstract—The aim of this paper is to answer the question: considering a finely heterogeneous medium submitted to some excitation, does an equivalent macroscopic description exist? An equivalent macroscopic description means here an intrinsic description, independent of the macroscopic boundary conditions. It is well known that the condition resides in a good separation of scales. This separation concerns both the structure of the medium and the excitation itself. The homogenization process using double scale asymptotic developments appears then to be the appropriate method giving the right answer to the question. This is emphasized in two simple examples.

1. INTRODUCTION

We consider an heterogeneous medium like for example a composite material, a fluid mixture or a porous media. We investigate the behaviour of a volume of such a medium, submitted to some excitation through boundary conditions or body forces. This is a classic boundary value problem of physical mathematics. When the number of heterogeneities is small (not too large) the solution can be deduced using analytical or numerical methods. Difficulties arise when the number of heterogeneities becomes too large (Fig. 1). A direct analytical approach becomes more and more intricated and the numerical formulation more and more unstable.

An old, classic idea is to replace the heterogeneous medium by a continuous, equivalent one which gives the average behaviour of the medium submitted to the excitation, at the macroscopic scale, i.e. at the scale of the volume containing a large number of heterogeneities. As a matter of fact, the existence of such equivalent media is the basic assumption when investigating them directly at the macroscopic scale, using experimental or phenomenological approaches. Another route to obtain macroscopic descriptions is to pass from the description at the heterogeneity scale (we will subsequently call it the microscopic scale or the local scale) to the macroscopic one. These processes are named homogenization processes. They are extremely numerous and fruitful, depending on the different mathematical techniques. Let us note among others the homogenization for fine periodic structures [1, 2], the statistical modelling [3], the self-consistent method (for example [4]), and generally speaking all methods using the average theorem [5–8].

Whatever the method employed, phenomenological investigation or homogenization process, it is a truism to say that the validity of the result (the so-called macroscopic equivalent description) suppose that the medium and the excitation are homogenizable, i.e. there exists an equivalent macroscopic description. As a matter of fact, this is frequently taken as a basic assumption. A frequent snare in this field is constituted by the average theorem, which is valid whether or not homogenization is possible.

The aim of this paper is to present the method to be followed in order to avoid similar problems. It will become apparent that the double scale method using asymptotic developments as introduced in [1, 2] in the case of periodic structures gives good results. The second part is devoted to general features and a presentation of the methodology. Afterwards we investigate two simple, classic examples, i.e. the slow permanent flow of a Newtonian fluid through rigid porous media (Part 3) and the dynamics of elastic composite materials (Part 4).

For the sake of simplicity we restrict ourselves to the situation where the microscopic description is given by piecewise continuous physics. And a large number of fundamental points are not examined here, e.g. the physical significance of macroscopic quantities, the macroscopic boundary conditions to be taken into account when solving problems at the macroscopic scale, etc. It is obvious that the homogenizability is not enough to insure effective results. For more details concerning these points, see [9].

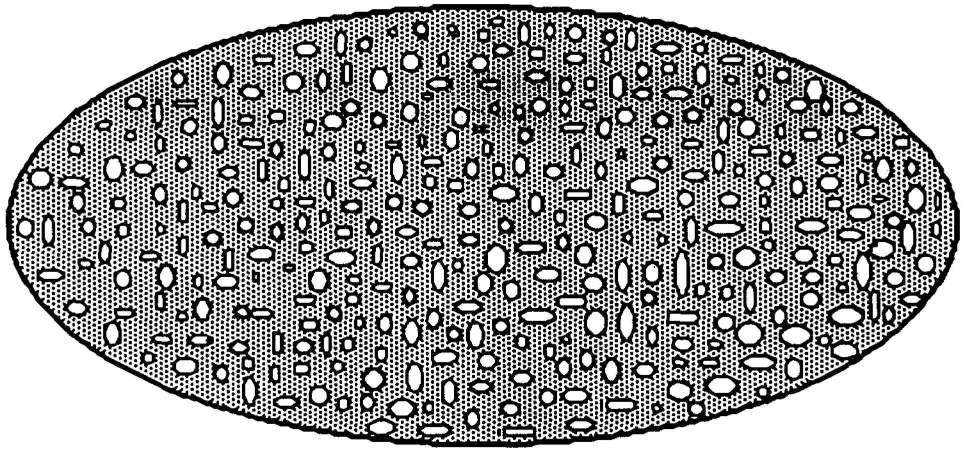

Fig. 1. Heterogeneous medium.

2. IS AN EQUIVALENT MACROSCOPIC DESCRIPTION POSSIBLE?

2.1 General

Before entering the subject we must give a more precise definition of what we are looking for when searching an equivalent macroscopic medium. We want to obtain an equivalent macroscopic boundary value problem, i.e. relations between macroscopic quantities (in practice "averaged" quantities, the meaning of which will be made subsequently clearer) and effective parameters. These relations consist of constitutive laws and balance equations, classic or not. The description is intrinsic to a large class of media submitted to a given excitation and thus must be independent of the macroscopic boundary conditions: the macroscopic description must be valid for every boundary value problem concerning a medium belonging to this class. The macroscopic description is continuous, as opposed to the microscopic one which is finely piecewise continuous. But it is clear that the macroscopic description is still valid for piecewise continuous macroscopic boundary value problems, on condition that the domains of continuity be such that homogenization is possible in each of them.

Let us now look for the conditions of homogenizability. It is clear from the discussion above that the macroscopic domain must contain a (very) large number of heterogeneities and the characteristic lengths of the heterogeneities must be limited so that there exists an elementary representative volume (ERV) of the medium, small compared to the macroscopic volume. The phenomena under investigation must also exhibit such an ERV. This is the definition of a good separation of scales which must be valid for both the geometry and the physical quantities. The ERV reduces to the periodic cell when the medium is periodic. Let l be a characteristic length of the ERV or the periodic cell and L a characteristic macroscopic length. L represents both a characteristic geometrical length of the volume of considered medium and a characteristic macroscopic length of the excitation. The right L is the smallest one. The separation of scales implies:

$$\frac{l}{L} = \varepsilon \ll 1 \tag{1}$$

It is important to notice that the physical quantities must also check the condition of separated scales. For example when considering a periodic elastic composite material satisfying (1) from the geometrical point of view, but submitted to a dynamical excitation with a wavelength $O(l)$, diffraction occurs. The L for the excitation is $O(l)$, and condition (1) is not fulfilled. We cannot obtain a macroscopic description which satisfies the conditions of homogenizability presented above (see Part 4). Diffraction shows stop-bands where the excitation is confined to boundary layers along the sources of excitation. Moreover in this case, random and periodic media do not behave in the same way.

Condition (1) is well admitted as the basic assumption for all the homogenization processes, even if most of them do not systematically make use of it. The two characteristic lengths l and L introduce two dimensionless space variables $\mathbf{X}/l$ and $\mathbf{X}/L$ where $\mathbf{X}$ is a physical space variable. Therefore due to the separation of scales each unknown Φ appears as a function of these two dimensionless space variables, among other variables. In practice it is more convenient to choose different (physical) space variables, i.e. $\mathbf{x}$ and $\mathbf{y} = \mathbf{x}\varepsilon$, where $\mathbf{x}$ is the macroscopic (or slow) space variable and $\mathbf{y}$ the microscopic (or fast) space variable. Two equivalent descriptions are then possible:

$$\Phi = \Phi(\mathbf{x}, \mathbf{y}), \qquad \mathbf{y} = \mathbf{x}/\varepsilon,$$

which is the macroscopic point of view,

$$\Phi = \Phi(\mathbf{x}, \mathbf{y}), \qquad \mathbf{x} = \varepsilon\mathbf{y},$$

which is the microscopic point of view.

Denote by $\langle\Phi\rangle$ the average of Φ. For random media the average is taken over an elementary representative volume, with respect to $\mathbf{y}$, whereas for periodic media the considered volume is the periodic cell. We have generally:

$$\Phi = O(\langle\Phi\rangle), \tag{2}$$

where the symbol $O(\cdot)$ must be understood with respect to ε:

$$\Phi = O(\langle\Phi\rangle) \quad \text{if} \quad \varepsilon \ll (\Phi/\langle\Phi\rangle) \ll \varepsilon^{-1}.$$

The separation of scales for Φ implies that (see Fig. 2):

$$\frac{\partial\Phi}{\partial\mathbf{y}} = O\left(\frac{\partial\langle\Phi\rangle}{\partial\mathbf{x}}\right) \tag{3}$$

i.e. the local gradient of Φ is of the same order of magnitude as the macroscopic gradient of $\langle\Phi\rangle$. From (2) we deduce:

$$\frac{\partial\Phi}{\partial\mathbf{y}} = O\left(\frac{\partial\Phi}{\partial\mathbf{x}}\right) \tag{4}$$

From (3) and Fig. 2 we see that the variation of $\langle\Phi\rangle$ against $\mathbf{x}$ over a length $O(l)$ is small, in fact zero in the limit case $\varepsilon \to 0$. It means that Φ checks some $\mathbf{y}$ stationariness property at the micro-scale. The $\mathbf{y}$ stationariness of Φ is defined as follows. Let $\langle\Phi\rangle$ and $\langle\Phi\rangle_s$ be the volume and surface average of Φ over an elementary representative volume and surface, respectively. The choice of either volume or surface average depends on the physical meaning of Φ. If Φ is for example a density, the volume average is the convenient choice. If Φ is a stress, then the surface has to be considered, etc. Φ is stationary if its average is invariant through a local translation, of order l. For example, for a surface average, let Σ_1 and Σ_2 be two parallel cross sections (or boundaries) of a parallelepipedic ERV, see Fig. 3. The condition is written:

$$\int_{\Sigma_1} \Phi \, \mathrm{d}S = \int_{\Sigma_2} \Phi \, \mathrm{d}S \tag{5}$$

A similar property holds when the microstructure is periodic, the elementary representative volume being replaced by the periodic cell. In this case Φ is $\mathbf{y}$ periodic.

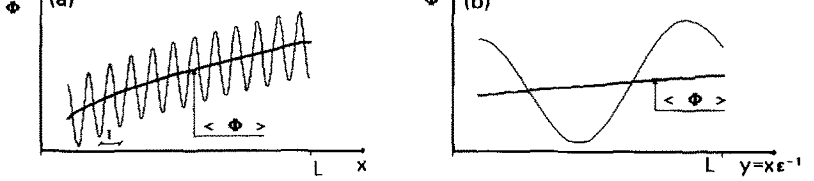

Fig. 2. Variations of a quantity Φ plotted against the two space variables x and y. (a) Macroscopic variations of Φ. (b) Local variations of Φ.

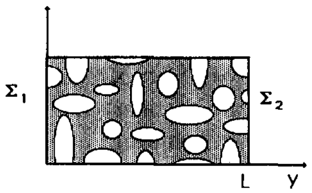

Fig. 3. Elementary representative volume of a random medium.

It is now obvious that the small parameter ε is the cornerstone of an homogenization process. This leads us to look for the unknowns in the form of asymptotic developments [1, 2]:

$$\Phi(\mathbf{x},\mathbf{y}) = \Phi^{(0)}(\mathbf{x},\mathbf{y}) + \varepsilon\Phi^{(1)}(\mathbf{x},\mathbf{y}) + \varepsilon^2\Phi^{(2)}(\mathbf{x},\mathbf{y}) + \cdots \tag{6}$$

Perfect homogenization is obtained when ε tends to zero.

The stationariness or the periodicity then implies the $\mathbf{y}$ stationariness or the $\mathbf{y}$ periodicity of the $\Phi^{(i)}$. This property in turn insures a good separation of scales and thus homogenizability: if the unknowns can be found in the form (6), where the $\Phi^{(i)}$ are $\mathbf{y}$-stationary or $\mathbf{y}$-periodic, then homogenization is possible. If not, the medium and the excitation under consideration are not homogenizable. The method is self-consistent.

2.2 *The methodology*

More precisely the method is as follows. We assume the local description to be given and we are looking for a scaled up equivalent description.

Firstly we adopt the macroscopic or the microscopic point of view. They are *a priori* equivalent, and the choice depends only on convenience. Consequently the asymptotics will be taken in the form (6) with $\mathbf{y} = \mathbf{x}/\varepsilon$ or $\mathbf{x} = \varepsilon\mathbf{y}$, respectively. $\mathbf{x}$ (respectively $\mathbf{y}$) is the driving space variable and L (respectively l) is the characteristic length to be used to make the various quantities involved in the description dimentionless.

Secondly we proceed with the normalization of the local description. This means that the local description is made dimensionless and the dimensionless numbers are evaluated with respect to the powers of ε. A quantity q is said to be $O(\varepsilon^p)$ if:

$$\varepsilon^{p+1} \ll q \ll \varepsilon^{p-1}$$

Normalization is an important step in which the physical background is fully taken into account. It is necessary before using asymptotics with powers of ε.

Thirdly we introduce asymptotic developments as (6) into the normalized local description, identify the like powers of ε and solve the succeeding cell problems thus obtained. This is nothing more than the process introduced in [1] for periodic media when using the macroscopic point of view. But now, due to the normalization, the first term of the asymptotic developments must be non zero:

$$\Phi^{(0)} \neq 0.$$

If we limit ourselves to the discovery of the structure of the macroscopic description, periodic and random media are equivalent, if homogenization is possible. It is therefore convenient to assume that the medium is periodic. The corner stone of homogenizability stands on a necessary and sufficient condition, often referred to as the compatibility condition, for the existence of the solutions for each cell problem. Equations to be solved are of the form:

$$\mathrm{div}_y\, \Phi^{(i)} + \mathrm{div}_x\, \Phi^{(i-1)} = 0,$$

where the subscripts x and y show derivatives with respect to $\mathbf{x}$ and $\mathbf{y}$, respectively, and div is the divergence operator. This is a (local) balance for $\Phi^{(i)}$ where $\mathrm{div}_x\, \Phi^{(i-1)}$ appears as a source term. Consequently, since the $\Phi^{(i)}$ are locally periodic or stationary, the source must be of zero average:

$$\langle \mathrm{div}_x\, \Phi^{(i-1)} \rangle = 0 \quad \text{or} \quad \mathrm{div}_x\, \langle \Phi^{(i-1)} \rangle = 0,$$

which leads either to the macroscopic description or to non-homogenizability when the compatibility condition leads to $\Phi^{(0)} = 0$.

The two following examples (Parts 3 and 4) emphasize the efficiency of the method. Following the above considerations the media are assumed locally periodic (but local stationariness would lead to identical results).

3. DARCY LAW

We consider a rigid porous periodic medium (Fig. 4). Ω_S is the solid part, Ω_l the pores occupied by a viscous incompressible newtonian fluid and Γ the boundary between the two. Ω_S and Ω_l are connected. We study the slow permanent flow of the fluid through the porous medium when it is submitted to a macroscopic gradient of pressure. This is a classic problem investigated by numerous authors and in particular in [10] when the porous medium is periodic. It has been established experimentally that the macroscopic equivalent description is the Darcy law:

$$\langle \mathbf{v} \rangle = -\mathbf{K}\,\mathbf{grad}\,p,$$

where $\langle \mathbf{v} \rangle$ is the macroscopic flux of fluid, $\mathbf{K}$ the permeability, **grad** is the gradient operator and p the pressure, with the volume balance:

$$\operatorname{div} \langle \mathbf{v} \rangle = 0.$$

At the microscopic level the fluid flow is given by:

—the Stokes equation,

$$\mu\,\Delta \mathbf{v} - \mathbf{grad}\,p = 0, \tag{7}$$

where Δ is the Laplacian operator.

—the incompressibility condition,

$$\operatorname{div} \mathbf{v} = 0, \tag{8}$$

—the adherence condition,

$$\mathbf{v} = 0 \quad \text{on} \quad \Gamma. \tag{9}$$

Here $\mathbf{v}$ is the local velocity, p the pressure, and μ the viscosity. Our aim is to show first that the method presented in Part 2 leads us to the right answer, and secondly how non-homogenizable situations are to be treated. We first recall the homogenization process of (7–9) and then investigate non-homogenizable problems.

3.1 Homogenization

Equation (7) introduces a dimensionless number $Q = \mathbf{grad}\,p/\mu\,\Delta\mathbf{v}$. It is obvious that before introducing into (7) such asymptotic developments as (6), we must measure Q with ε and then normalize (7). We look here for an homogenizable problem. So we consider an experimental verification of the Darcy law. The fluid is forced through the porous medium by a macroscopic pressure gradient. Then in equation (7):

$$\mathbf{grad}\,p = O(p/L),$$

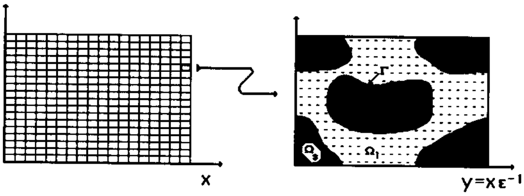

Fig. 4. Porous saturated medium.

where L is the size of the examined sample. In the same time the velocity $\mathbf{v}$ varies in the pores:

$$\mu\,\Delta\mathbf{v} = O(\mu v/l^2).$$

From equation (7) these two terms are of the same order:

$$\mu v/l^2 = O(p/L). \tag{10}$$

In the following we adopt the microscopic point of view. $\mathbf{v}$ and p are looked for in the form:

$$\begin{aligned}\mathbf{v}(\mathbf{x},\mathbf{y}) &= \mathbf{v}^{(0)}(\mathbf{x},\mathbf{y}) + \varepsilon\mathbf{v}^{(1)}(\mathbf{x},\mathbf{y}) + \varepsilon^2\mathbf{v}^{(2)}(\mathbf{x},\mathbf{y}) + \cdots\\ p(\mathbf{x},\mathbf{y}) &= p^{(0)}(\mathbf{x},\mathbf{y}) + \varepsilon p^{(1)}(\mathbf{x},\mathbf{y}) + \varepsilon^2 p^{(2)}(\mathbf{x},\mathbf{y}) + \cdots\end{aligned} \tag{11}$$

with $\mathbf{x} = \varepsilon\mathbf{y}$, $\mathbf{v}^{(i)}$ and $p^{(i)}$ periodic in $\mathbf{y}$.

Therefore the characteristic length to normalize equation (7) is l:

$$Q = \mathbf{grad}\,p/\mu\,\Delta\mathbf{v} = O(pl^2/l\mu v),$$

and from (10): $Q = O(\varepsilon^{-1})$.

We then formally normalize equation (7) by:

$$\varepsilon\mu\,\Delta\mathbf{v} - \mathbf{grad}\,p = 0 \tag{12}$$

We are now ready to perform the homogenization process. We introduce the developments (11) in (12, 8, 9), noticing that the operator **grad** becomes $\mathbf{grad}_y + \varepsilon\,\mathbf{grad}_x$, where the subscripts x and y show derivatives with respect to $\mathbf{x}$ and $\mathbf{y}$, respectively. We obtain for the different powers of ε, using (4):

$$\begin{aligned}\mathbf{grad}_y\,p^{(0)} &= 0,\\ \mu\,\Delta_y\mathbf{v}^{(0)} - \mathbf{grad}_x\,p^{(0)} - \mathbf{grad}_y\,p^{(1)} &= 0,\end{aligned} \tag{13}$$

$$\begin{aligned}\mathrm{div}_y\,\mathbf{v}^{(0)} &= 0,\\ \mathrm{div}_y\,\mathbf{v}^{(1)} + \mathrm{div}_x\,\mathbf{v}^{(0)} &= 0,\end{aligned} \tag{14}$$

$$\begin{aligned}\mathbf{v}^{(0)} &= 0,\\ \mathbf{v}^{(1)} &= 0, \quad \text{on} \quad \Gamma\end{aligned} \tag{15}$$

Equation $(13)_1$ gives $p^{(0)} = p^{(0)}(\mathbf{x})$.

Next, $(13)_2$ with $(14)_1$ and $(15)_1$ represents the basic cell problem. It shows that $\mathbf{v}^{(0)}$ and $p^{(1)}$ are linear functions of $\mathbf{grad}_x\,p^{(0)}$ (for more details see [10]). In particular:

$$v_i^{(0)} = -k_{ij}\frac{\mathrm{d}p^{(0)}}{\mathrm{d}x_j}.$$

Now consider equation $(14)_2$. It is a local balance volume for $\mathbf{v}^{(1)}$ where $\mathrm{div}_x\,\mathbf{v}^{(0)}$ appears as a source term. Furthermore $\mathbf{v}^{(1)}$ is Ω periodic and from $(15)_2$ it is zero valued on Γ. Therefore the source term $\mathrm{div}_x\,\mathbf{v}^{(0)}$ must check a compatibility condition: its average must be null. This can be seen integrating $(14)_2$ over Ω_l. Putting:

$$\langle * \rangle = |\Omega|^{-1}\int_{\Omega_l} * \,\mathrm{d}\Omega$$

we obtain:

$$\langle \mathrm{div}_x\,\mathbf{v}^{(0)}\rangle = \mathrm{div}_x\,\langle\mathbf{v}^{(0)}\rangle = -|\Omega|^{-1}\int_{\Omega_l}\mathrm{div}_y\,\mathbf{v}^{(1)}\,\mathrm{d}\Omega = -|\Omega|^{-1}\int_{\delta\Omega_l}\mathbf{v}^{(1)}\mathbf{N}\,\mathrm{d}S = 0. \tag{16}$$

where $\mathbf{N}$ is a unit exterior normal to Γ. It is easy to see that this result is also valid if stationariness is present, by using relations similar to (5).

Finally we have:

$$\mathrm{div}_x\,\langle\mathbf{v}^{(0)}\rangle = 0, \qquad \langle\mathbf{v}^{(0)}\rangle = -\mathbf{K}\,\mathbf{grad}_x\,p^{(0)}, \qquad \mathbf{K} = \langle\mathbf{k}\rangle, \tag{17}$$

which is the announced macroscopic volume balance and Darcy law (on condition that the volume average $\langle\mathbf{v}^{(0)}\rangle$ be a flux, i.e. a surface average, see [9]).

It should be noted that the local periodicity of the unknowns (which a direct consequence of the homogenizability) is the basis of the result. Dropping it, (16) is no longer valid and (17) cannot be obtained. Using relations similar to (5), a local stationariness (for a random porous medium) leads to an identical result (16) where Ω is now the elementary representative volume. If we admit the uniqueness of the basic problem $(13)_2$, $(14)_1$ and $(15)_1$, (17) follows.

3.2 Non-homogenizable situation

If we consider $Q = pl/\mu v > \varepsilon^{-1}$, the problem is not homogenizable: the forcing term $\mathbf{grad}\, p$ is larger and the separation of scales weakened. Consider for example $Q = O(\varepsilon^{-2})$. Thus equation (7) is now normalized by:

$$\varepsilon^2 \mu\, \Delta \mathbf{v} - \mathbf{grad}\, p = 0,$$

which gives for the different orders:

$$\begin{aligned} \mathbf{grad}_y\, p^{(0)} &= 0, \\ \mathbf{grad}_x\, p^{(0)} + \mathbf{grad}_y\, p^{(1)} &= 0, \\ \mu\, \Delta_y \mathbf{v}^{(0)} - \mathbf{grad}_x\, p^{(1)} - \mathbf{grad}_y\, p^{(2)} &= 0 \end{aligned} \tag{18}$$

Equation $(18)_1$ gives again $p^{(0)} = p^{(0)}(\mathbf{x})$. But $(18)_2$ with $p^{(1)}$ Ω periodic introduces the compatibility condition $\mathbf{grad}_x\, p^{(0)} = 0$. Thus $p^{(0)}$ is $\mathbf{x}$ independent and $(18)_2$ leads to $p^{(1)} = p^{(1)}(\mathbf{x})$. Finally $(18)_3$ gives $\mathbf{v}^{(0)}$ as a linear vectorial function of $\mathbf{grad}_x\, p^{(1)}$. Regrouping $p^{(0)}$ and $p^{(1)}$ in $p^{(0)} + \varepsilon p^{(1)}$ with

$$\mathbf{grad}_x\, (p^{(0)} + \varepsilon p^{(1)}) = \varepsilon\, \mathbf{grad}_x\, p^{(1)},$$

leads as above to the Darcy law with a smaller macroscopic pressure gradient, with $Q = O(\varepsilon^{-1})$.

Consider now cases where $Q < \varepsilon^{-1}$. Let us investigate for example the case where $Q = O(1)$. Equation (7) is not invariant under normalization. And the set (7–9) gives for different orders:

$$\begin{aligned} \mu\, \Delta_y\, \mathbf{v}^{(0)} - \mathbf{grad}_y\, p^{(0)} &= 0, \\ \mu\, \Delta_y \mathbf{v}^{(1)} + \mu(\mathrm{div}_x\, \mathbf{grad}_y\, \mathbf{v}^{(0)} + \mathrm{div}_y\, \mathbf{grad}_x\, \mathbf{v}^{(0)}) - \mathbf{grad}_x\, p^{(0)} - \mathbf{grad}_y\, p^{(1)} &= 0 \end{aligned} \tag{19}$$

$$\begin{aligned} \mathrm{div}_y\, \mathbf{v}^{(0)} &= 0, \\ \mathrm{div}_x\, \mathbf{v}^{(0)} + \mathrm{div}_y\, \mathbf{v}^{(1)} &= 0, \\ \mathrm{div}_x\, \mathbf{v}^{(1)} + \mathrm{div}_y\, \mathbf{v}^{(2)} &= 0, \end{aligned} \tag{20}$$

$$\begin{aligned} &\mathbf{v}^{(0)} = 0, \\ &\mathbf{v}^{(1)} = 0, \\ &\mathbf{v}^{(2)} = 0, \quad \text{on} \quad \Gamma. \end{aligned} \tag{21}$$

The set $(19)_1$, $(20)_1$, $(21)_1$, with $\mathbf{v}^{(0)}$ and $p^{(0)}\Omega$ periodic, represents an homogeneous boundary value problem. It is easy to show the uniqueness of the solution, as in Part 3.1. Therefore the solution is:

$$\mathbf{v}^{(0)} = 0, \qquad p^{(0)} = p^{(0)}(\mathbf{x}).$$

The next problem concerns $\mathbf{v}^{(1)}$ and $p^{(1)}$ and is written:

$$\begin{aligned} \mu\, \Delta_y \mathbf{v}^{(1)} - \mathbf{grad}_x\, p^{(0)} - \mathbf{grad}_y\, p^{(1)} &= 0, \\ \mathrm{div}_y\, \mathbf{v}^{(1)} &= 0, \\ \mathbf{v}^{(1)} &= 0, \quad \text{on} \quad \Gamma. \end{aligned}$$

It is the boundary value problem (the basic cell problem) investigated in Part 3.1, where $\mathbf{v}^{(0)}$ is replaced by $\mathbf{v}^{(1)}$. Therefore:

$$\mathbf{v}^{(1)} = -\mathbf{k}\, \mathbf{grad}_x\, p^{(0)}.$$

Finally $(20)_3$ with $(21)_3$ and the Ω periodicity gives the compatibility condition:

$$\mathrm{div}_x \langle \mathbf{v}^{(1)} \rangle = 0 \quad \text{with} \quad \langle \mathbf{v}^{(1)} \rangle = -\mathbf{K}\, \mathbf{grad}_x\, p^{(0)}.$$

As we see we again obtain an homogenized macroscopic description. But the first non-zero term for $\mathbf{v}$ is now $\varepsilon\mathbf{v}^{(1)}$ so that Q is *ipso facto* increased to $O(\varepsilon^{-1})$. A situation with $Q = O(1)$ can actually occur but is not homogenizable. An intrinsic description is not possible. Looking for such a description by using for example the average theorem would lead to a "macroscopic" description depending directly upon the boundary values on the external boundary of the medium. The method used here is self-consistent: it only gives results where homogenization is possible.

3.3 Conclusion

Determining an equivalent macroscopic description as defined in Part 2 to the permanent flow of a newtonian incompressible flow through a rigid porous media is possible only if $Q = pl/\mu v = \varepsilon^{-1}$.

4. DYNAMICS OF ELASTIC COMPOSITE MATERIALS

We consider in this part a periodic elastic composite material made of two different elastic components Ω_1 and Ω_2 with a common boundary Γ (Fig. 5). Here again a random medium with local stationariness would lead to similar results.

The local description is given by:

$$\mathrm{div}(\mathbf{ae}(\mathbf{u})) = \rho\frac{\partial^2\mathbf{u}}{\partial t^2} \tag{22}$$

where $\mathbf{e}$ is the small deformation, $\mathbf{u}$ the displacement, ρ the density and $\mathbf{a}$ an elastic tensor which takes the values $\mathbf{a}_1$ in Ω_1 and $\mathbf{a}_2$ in Ω_2, respectively. We assume $\mathbf{a}_1$ and $\mathbf{a}_2$ to be of the same order of magnitude:

$$\mathbf{a}_1 = O(\mathbf{a}_2).$$

On Γ we have the classic conditions:

$$[\mathbf{u}] = 0, \tag{23}$$

$$[\mathbf{ae}(\mathbf{u})\mathbf{N}] = 0. \tag{24}$$

where $\mathbf{N}$ is a unit normal to Γ and $[\Phi]$ denotes the discontinuity of Φ on Γ.

4.1 Homogenization

Homogenization is possible if the scales are well separated. In a dynamic problem a good candidate for the macroscopic length L is the wavelength λ. Therefore we must have $l/\lambda = \varepsilon \ll 1$ where l is the size of the periodic cell. As $\mathbf{a}_1$ and $\mathbf{a}_2$ are of the same order of magnitude, we may anticipate that the effective elastic tensor will be of the same order. Introducing a characteristic time T (the period) we have classically:

$$L^2 = \lambda^2 = O(\mathbf{a}T^2/\rho). \tag{25}$$

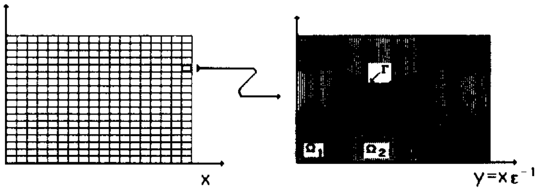

Fig. 5. Composite material.

Now equation (22) introduces a dimensionless number P, which is written when using l to normalize:

$$P = \rho\frac{\partial^2\mathbf{u}}{\partial t^2}\Big/\mathrm{div}(\mathbf{ae}(\mathbf{u})) = O\left(\frac{\rho l^2}{aT^2}\right),$$

and with (25):

$$P = O(l^2/L^2) = O(\varepsilon^2).$$

Thus we formally normalize (22) in the form:

$$\mathrm{div}(\mathbf{ae}(\mathbf{u})) = \varepsilon^2\rho\frac{\partial^2\mathbf{u}}{\partial t^2} \tag{26}$$

Introducing in (26), (23) and (24) an asymptotic development for $\mathbf{u}$ in the form:

$$\mathbf{u}(\mathbf{x},\mathbf{y}) = \mathbf{u}^{(0)}(\mathbf{x},\mathbf{y}) + \varepsilon\mathbf{u}^{(1)}(\mathbf{x},\mathbf{y}) + \varepsilon^2\mathbf{u}^{(2)}(\mathbf{x},\mathbf{y}) + \cdots$$

with $\mathbf{x} = \varepsilon\mathbf{y}$, and $\mathbf{u}^{(i)}$ Ω periodic, it becomes successively:

$$\begin{aligned}\mathrm{div}_y(\mathbf{ae}(\mathbf{u}^{(0)})) &= 0,\\ \mathrm{div}_y(\mathbf{a}(\mathbf{e}_y(\mathbf{u}^{(1)}) + \mathbf{e}_x(\mathbf{u}^{(0)}))) + \mathrm{div}_x(\mathbf{ae}_{yy}(\mathbf{u}^{(0)})) &= 0.\\ \mathrm{div}_y(\mathbf{a}(\mathbf{e}_\mathbf{y}(\mathbf{u}^{(2)}) + \mathbf{e}_\mathbf{x}(\mathbf{u}^{(1)}))) + \mathrm{div}_x(\mathbf{a}(\mathbf{e}_\mathbf{y}(\mathbf{u}^{(1)}) + \mathbf{e}_\mathbf{x}(\mathbf{u}^{(0)}))) &= \rho\frac{\partial^2\mathbf{u}^{(0)}}{\partial t^2},\end{aligned} \tag{27}$$

$$\begin{aligned}&[\mathbf{u}^{(0)}] = 0,\\ &[\mathbf{u}^{(1)}] = 0,\\ &[\mathbf{u}^{(2)}] = 0, \quad \text{on} \quad \Gamma\end{aligned} \tag{28}$$

$$\begin{aligned}[\mathbf{ae}_y(\mathbf{u}^{(0)})\mathbf{N}] &= 0,\\ [\mathbf{a}(\mathbf{e}_y(\mathbf{u}^{(1)}) + \mathbf{e}_x(\mathbf{u}^{(0)}))\mathbf{N}] &= 0,\\ [\mathbf{a}(\mathbf{e}_y(\mathbf{u}^{(2)}) + \mathbf{e}_x(\mathbf{u}^{(1)}))\mathbf{N}] &= 0 \quad \text{on} \quad \Gamma.\end{aligned} \tag{29}$$

The set $(27)_1$, $(28)_1$ and $(29)_1$ is an homogeneous set with solution:

$$\mathbf{u}^{(0)} = \mathbf{u}^{(0)}(\mathbf{x}, t).$$

Then the next problem concerns $\mathbf{u}^{(1)}$. It is the basic cell problem given by $(27)_2$, $(28)_2$ and $(29)_2$. $\mathbf{u}^{(1)}$ appears as linearly depending on $\mathbf{e}_x(\mathbf{u}^{(0)})$, to an additive arbitrary constant $\bar{\mathbf{u}}^{(1)}(\mathbf{x}, t)$ with respect to $\mathbf{y}$:

$$\mathbf{u}^{(1)} = \boldsymbol{\xi}\mathbf{e}_\mathbf{x}(\mathbf{u}^{(0)}) + \bar{\mathbf{u}}^{(1)}(\mathbf{x}, t),$$

where $\boldsymbol{\xi}$ is a third order tensor, $\mathbf{y}$ depending.

Finally $(27)_3$ is a momentum balance with a source term:

$$\mathrm{div}_x(\mathbf{a}(\mathbf{e}_\mathbf{y}(\mathbf{u}^{(1)}) + \mathbf{e}_\mathbf{x}(\mathbf{u}^{(0)}))) - \rho\frac{\partial^2\mathbf{u}^{(0)}}{\partial t^2}$$

In this way it introduces a compatibility condition which is obtained by integrating $(27)_3$ over Ω_1 and Ω_2, respectively, and by using $(29)_3$. The average of the source term is shown to cancel:

$$\langle\mathrm{div}_x(\mathbf{a}(\mathbf{e}_\mathbf{y}(\mathbf{u}^{(1)}) + \mathbf{e}_\mathbf{x}(\mathbf{u}^{(0)})))\rangle - \langle\rho\rangle\frac{\partial^2\mathbf{u}^{(0)}}{\partial t^2} = 0,$$

or

$$\mathrm{div}_x(\mathbf{c}\mathbf{e}_\mathbf{x}(\mathbf{u}^{(0)})) - \langle\rho\rangle\frac{\partial^2\mathbf{u}^{(0)}}{\partial t^2} = 0, \tag{30}$$

where

$$\langle * \rangle = |\Omega|^{-1}\int_\Omega * \,\mathrm{d}\Omega,$$

$\mathbf{c} = \langle \mathbf{a}\mathbf{e}_y(\boldsymbol{\xi} + \mathbf{a}) \rangle$ is the effective elastic tensor and (30) represents the macroscopic equivalent description at the first order of magnitude.

It is obvious that a situation in which $P = O(\varepsilon^p)$, $p > 2$ are also homogenizable. Following the same route we obtain:

$$\mathrm{div}_x(\mathbf{c}\mathbf{e}_x(\mathbf{u}^{(0)})) = 0.$$

In these cases, the dynamics reduces at the first order to statics. This was to be expected since the inertial term

$$\langle \rho \rangle \frac{\partial^2 \mathbf{u}^{(0)}}{\partial t^2}$$

becomes negligible.

4.2 Non-homogenizable situation

Let us now consider P of order ε^p, $p < 2$, say for example $P = O(\varepsilon)$. Therefore (22) is now normalized in the form:

$$\mathrm{div}(\mathbf{a}\mathbf{e}(\mathbf{u})) = \varepsilon \rho \frac{\partial^2 \mathbf{u}^{(0)}}{\partial t^2}$$

We have again $\mathbf{u}^{(0)} = \mathbf{u}^{(0)}(\mathbf{x}, t)$. But the cell problem giving $\mathbf{u}^{(1)}$ is now written:

$$\mathrm{div}_y(\mathbf{a}(\mathbf{e}_y(\mathbf{u}^{(1)}) + \mathbf{e}_x(\mathbf{u}^{(0)}))) = \rho \frac{\partial^2 \mathbf{u}^{(0)}}{\partial t^2}$$

$$[\mathbf{u}^{(1)}] = 0,$$

$$[\mathbf{a}(\mathbf{e}_y(\mathbf{u}^{(1)}) + \mathbf{e}_x(\mathbf{u}^{(0)}))\mathbf{N}] = 0 \quad \text{on} \quad \Gamma.$$

This set admits a solution on the condition that a compatibility relation is checked. This one is obtained following the same method as above:

$$\langle \rho \rangle \frac{\partial^2 \mathbf{u}^{(0)}}{\partial t^2} = 0,$$

and since $\langle \rho \rangle \neq 0$:

$$\frac{\partial^2 \mathbf{u}^{(0)}}{\partial t^2} = 0.$$

Thus *ipso facto* P is decreased to ε^2 where an homogenization is possible. It is again shown that the method is self-consistent. Situations where $P = O(\varepsilon^p)$, $p < 2$ most certainly exist but they are not homogenizable. In the case examined above, i.e. $P = O(\varepsilon)$, the wavelength λ is $O(L\sqrt{\varepsilon}) \ll L$, there is no separation of scales. Diffraction occurs, as discussed in Part 2, which concerns boundary macroscopic layers along the excitation sources and thus does not belong to the class of homogenizable situations.

4.3 Conclusion

Determining an equivalent macroscopic description, as defined in Part 2, to propagation of waves through an elastic composite material is only possible if $P = \rho l^2/\mathbf{a}T^2 \leq \varepsilon^2$. If $P < \varepsilon^2$ the behaviour is static at the first order of magnitude.

5. CONCLUSION

The process using double scale asymptotic developments appears as an heuristic method giving the right answer when we are looking for a possible macroscopic description equivalent to the behaviour of a finely heterogeneous medium submitted to some excitation. It introduces a basic cell problem where the cell is either the period when the medium is periodic or the elementary representative volume when it is random. It gives the general structure of the macroscopic description.

We may therefore recommend it, specially for intricate problems. Subsequently the other homogenization processes can be used to obtain the effective parameters, or bounds for them. The statistical modelling developed by Kröner [3] is a good candidate here, even if the author restricts its application to the statics: it is likely that, following the indicated route, the process could be shown to apply to the dynamics of elastic composite materials.

REFERENCES

[1] A. BENSOUSSAN, J. L. LIONS and G. C. PAPANICOLAOU, *Asymptotic Analysis for Periodic Structure.* North-Holland, Amsterdam (1978).
[2] E. SANCHEZ-PALENCIA, *Non-Homogeneous Media and Vibration Theory. Lecture Notes in Physics,* Vol. 127, Springer, Berlin (1980).
[3] E. KRÖNER, in *Modelling Small Deformations of Polycrystals,* Chap. 8 (Edited by J. GITTUS and J. ZARKA). Elsevier, London (1986).
[4] A. ZAOUI, in *Homogenization Techniques for Composite Media. Lecture Notes in Physics,* Vol. 272. Springer, Berlin (1987).
[5] R. I. NIGMATULIN, *Fluid Mech.* **10** (4), 72 (1981).
[6] A. BEDFORD and D. S. DRUMHELLER, *Int. J. Engng Sci.* **21** (8), 863 (1983).
[7] F. HOWES and S. WHITAKERS, *Chem. Engng Sci.* **40** (8), 1387 (1985).
[8] F. GILBERT, in *Manuel de Rheologie des Geomateriaux,* p. 317. Presses des Ponts et Chaussées (1987).
[9] J. L. AURIAULT and D. CAILLERIE, *Rev. Fr. Géotech.* **49,** 43 (1989).
[10] H. I. ENE and E. SANCHEZ-PALENCIA, *J. Méc.* **14** (1), 73 (1975).

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